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CALDERÓN INVERSE PROBLEM WITH PARTIAL DATA ON RIEMANN SURFACES

COLIN GUILLARMOU AND LEO TZOU

ABSTRACT. On a fixed smooth compact Riemann surface with boundary (M_0, g) , we show that for the Schrödinger operator $\Delta + V$ with potential $V \in C^{1,\alpha}(M_0)$ for some $\alpha > 0$, the Dirichlet-to-Neumann map $\mathcal{N}|_\Gamma$ measured on an open set $\Gamma \subset \partial M_0$ determines uniquely the potential V . We also discuss briefly the corresponding consequences for potential scattering at 0 frequency on Riemann surfaces with asymptotically Euclidean or asymptotically hyperbolic ends.

1. INTRODUCTION

The problem of determining the potential in the Schrödinger operator by boundary measurement goes back to Calderón [8]. Mathematically, it amounts to ask if one can detect some data from boundary measurement in a domain (or manifold) Ω with boundary. The typical model to have in mind is the Schrödinger operator $P := \Delta_g + V$ where g is a metric and V a potential, then we define the Cauchy data space by

$$\mathcal{C} := \{(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}) ; u \in H^1(\Omega), u \in \ker P\}$$

where ∂_ν is the interior pointing normal vector field to $\partial\Omega$.

The first natural question is the following *full data* inverse problem: does the Cauchy data space determine uniquely the metric g and/or the potential V ? In a sense, the most satisfying known results are when the domain $\Omega \subset \mathbb{R}^n$ is already known and g is the Euclidean metric, then the recovery of V has been proved in dimension $n > 2$ by Sylvester-Uhlmann [28], and very recently in dimension 2 by Bukhgeim [6] when the domain is simply connected. A related question is the conductivity problem which consists in taking $V = 0$ and replacing Δ_g by $-\operatorname{div} \sigma \nabla$ where σ is a definite positive symmetric tensor. An elementary observation shows that the problem of recovering an sufficiently smooth isotropic conductivity (i.e. $\sigma = \sigma_0 \operatorname{Id}$ for a function σ_0) is contained in the problem above of recovering a potential V . For domain of \mathbb{R}^2 , Nachman [26] used the $\bar{\partial}$ techniques to show that the Cauchy data space determines the conductivity. Recently a new approach developed by Astala and Päiväranta in [2] improved this result to assuming that the conductivity is only a L^∞ scalar function. This was later generalized to L^∞ anisotropic conductivities by Astala-Lassas-Päiväranta in [3]. We notice that there still are rather few results in the direction of recovering the Riemannian manifold (Ω, g) when $V = 0$, for instance the surface case by Lassas-Uhlmann [21] (see also [4, 15]), the real-analytic manifold case by Lassas-Taylor-Uhlmann [20] (see also [14] for the Einstein case), the case of manifolds admitting limiting Carleman weights and in a same conformal class by Dos Santos Ferreira-Kenig-Salo-Uhlmann [9].

The second natural, but harder, problem is the *partial data* inverse problem: if Γ_1 and Γ_2 are open subsets of $\partial\Omega$, does the partial Cauchy data space for P

$$\mathcal{C}_{\Gamma_1, \Gamma_2} := \{(u|_{\Gamma_1}, \partial_\nu u|_{\Gamma_2}); u \in H^1(\partial M_0), Pu = 0, u = 0 \text{ in } \partial\Omega \setminus \Gamma_1\}$$

determine the domain Ω , the metric, the potential? For a fixed domain of \mathbb{R}^n , the recovery of the potential if $n > 2$ with partial data measurements was initiated by Bukhgeim-Uhlmann [7] and later improved by Kenig-Sjöstrand-Uhlmann [18] to the case where Γ_1 and Γ_2 are respectively open subsets of the "front" and "back" ends of the domain. We refer the reader to the references for a more precise formulation of the problem. In dimension 2, the recent works of Imanuvilov-Uhlmann-Yamamoto [16] solves the problem for fixed domains Ω of \mathbb{R}^2 in the case when $\Gamma_1 = \Gamma_2$ and when the potential are in $C^{2,\alpha}(\Omega)$ for some $\alpha > 0$.

In this work, we address the same question when the background domain is a fixed Riemann surface with boundary. We prove the following recovery result:

Theorem 1.1. *Let (M_0, g) be a smooth compact Riemann surface with boundary and let Δ_g be its positive Laplacian. For $\alpha \in (0, 1)$, let $V_1, V_2 \in C^{1,\alpha}(M_0)$ be two real potentials and, for $i = 1, 2$, let*

$$(1) \quad \mathcal{C}_i^\Gamma =: \{(u|_\Gamma, \partial_\nu u|_\Gamma); u \in H^1(M_0), (\Delta_g + V_i)u = 0, u = 0 \text{ on } \partial M_0 \setminus \Gamma\}$$

be the respective Cauchy partial data spaces. If $\mathcal{C}_1^\Gamma = \mathcal{C}_2^\Gamma$ then $V_1 = V_2$.

Here the space $C^{1,\alpha}(M_0)$ is the usual Hölder space for $\alpha \in (0, 1)$. Notice that when $\Delta_g + V_i$ do not have L^2 eigenvalues for the Dirichlet condition, the statement above can be given in terms of Dirichlet-to-Neumann operators. Since $\Delta_{\hat{g}} = e^{-2\varphi} \Delta_g$ when $\hat{g} = e^{2\varphi} g$ for some function φ , it is clear that in the statement in Theorem 1.1, we only need to fix the conformal class of g instead of the metric g (or equivalently to fix the complex structure on M_0). In particular, the smoothness assumption of the Riemann surface with boundary is not really essential since we can change it conformally to make it smooth and for the Cauchy data space, this just has the effect of changing the potential conformally (we only need this new potential to be $C^{1,\alpha}$). Observe also that Theorem 1.1 implies that, for a fixed Riemann surface with boundary (M_0, g) , the Dirichlet-to-Neumann map on Γ for the operator $u \rightarrow -\text{div}_g(\gamma \nabla^g u)$ determines the isotropic conductivity γ if $\gamma \in C^{3,\alpha}(M_0)$ in the sense that two conductivities giving rise to the same Dirichlet-to-Neumann are equal. This is a standard observation by transforming the conductivity problem to a potential problem with potential $V := (\Delta_g \gamma^{\frac{1}{2}})/\gamma^{\frac{1}{2}}$. So our result also extends that of Henkin-Michel [15] in the case of isotropic conductivities.

The method to reconstruct the potential follows [6, 16] and is based on the construction of a large set of special complex geometric optic solutions of $(\Delta_g + V)u = 0$, more precisely if $\Gamma_0 = \partial M_0 \setminus \Gamma$ is the set where we do not know the Dirichlet-to-Neumann operator, we construct solutions of the form $u = \text{Re}(e^{\Phi/h}(a + r(h))) + e^{\text{Re}(\varphi)/h} s(h)$ with $u|_{\Gamma_0} = 0$ where $h > 0$ is a small parameter, Φ and a are holomorphic functions on (M_0, g) , independent of h , $\|r(h)\|_{L^2} = O(h)$ while $\|s(h)\|_{L^2} = O(h^{3/2} |\log h|)$ as $h \rightarrow 0$. The idea of [6] to reconstruct $V(p)$ for $p \in M_0$ is to take Φ with a non-degenerate critical point at p and then use stationary phase as $h \rightarrow 0$. In our setting, the function Φ needs to be purely real on Γ_0 an Morse with a prescribed critical point at p . One of our main contribution is a geometric construction of the holomorphic Carleman weights Φ satisfying such conditions. We should point out that we use a quite different method than in [16] to construct this weight, and we believe that our

method simplifies their construction even in their case. A Carleman estimate on the surface for this degenerate weight needs to be proved, and we follow ideas of [16]. We manage to improve the regularity of the potential to $C^{1,\alpha}$ instead of $C^{2,\alpha}$ in [16]. We finally notice that we provide a proof, in an Appendix, of the fact that the partial Cauchy data space \mathcal{C}^Γ determine a potential $V \in C^{0,\alpha}(M_0)$ on Γ (for $\alpha > 0$).

In Section 6, we obtain two inverse scattering results as corollary of Theorem 1.1: first for partial data scattering at 0 frequency for $\Delta + V$ on asymptotically hyperbolic surfaces with potential decaying at the conformal infinity, and secondly for full data scattering at 0 frequency for $\Delta + V$ with V compactly supported on an asymptotically Euclidean surface.

Another straightforward corollary in the asymptotically Euclidean case full data setting is the recovery of a compactly supported potential from the scattering operator at a positive frequency. The proof is essentially the same as for the operator $\Delta_{\mathbb{R}^n} + V$ once we know Theorem (1.1), so we omit it.

2. HARMONIC AND HOLOMORPHIC MORSE FUNCTIONS ON A RIEMANN SURFACE

2.1. Riemann surfaces. We start by recalling few elementary definitions and results about Riemann surfaces, see for instance [10] for more details. Let (M_0, g_0) be a compact connected smooth Riemannian surface with boundary ∂M_0 . The surface M_0 can be considered as a subset of a compact Riemannian surface (M, g) , for instance by taking the double of M_0 and extending smoothly the metric g_0 to M . The conformal class of g on the closed surface M induces a structure of closed Riemann surface, i.e. a closed surface equipped with a complex structure via holomorphic charts $z_\alpha : U_\alpha \rightarrow \mathbb{C}$. The Hodge star operator \star acts on the cotangent bundle T^*M , its eigenvalues are $\pm i$ and the respective eigenspace $T_{1,0}^*M := \ker(\star + i\text{Id})$ and $T_{0,1}^*M := \ker(\star - i\text{Id})$ are sub-bundle of the complexified cotangent bundle $\mathbb{C}T^*M$ and the splitting $\mathbb{C}T^*M = T_{1,0}^*M \oplus T_{0,1}^*M$ holds as complex vector spaces. Since \star is conformally invariant on 1-forms on M , the complex structure depends only on the conformal class of g . In holomorphic coordinates $z = x + iy$ in a chart U_α , one has $\star(udx + vdy) = -vdx + udy$ and

$$T_{1,0}^*M|_{U_\alpha} \simeq \mathbb{C}dz, \quad T_{0,1}^*M|_{U_\alpha} \simeq \mathbb{C}d\bar{z}$$

where $dz = dx + idy$ and $d\bar{z} = dx - idy$. We define the natural projections induced by the splitting of $\mathbb{C}T^*M$

$$\pi_{1,0} : \mathbb{C}T^*M \rightarrow T_{1,0}^*M, \quad \pi_{0,1} : \mathbb{C}T^*M \rightarrow T_{0,1}^*M.$$

The exterior derivative d defines the De Rham complex $0 \rightarrow \Lambda^0 \rightarrow \Lambda^1 \rightarrow \Lambda^2 \rightarrow 0$ where $\Lambda^k := \Lambda^k T^*M$ denotes the real bundle of k -forms on M . Let us denote $\mathbb{C}\Lambda^k$ the complexification of Λ^k , then the ∂ and $\bar{\partial}$ operators can be defined as differential operators $\partial : \mathbb{C}\Lambda^0 \rightarrow T_{1,0}^*M$ and $\bar{\partial} : \mathbb{C}\Lambda^0 \rightarrow T_{0,1}^*M$ by

$$(2) \quad \partial f := \pi_{1,0} df, \quad \bar{\partial} f := \pi_{0,1} df,$$

they satisfy $d = \partial + \bar{\partial}$ and are expressed in holomorphic coordinates by

$$\partial f = \partial_z f dz, \quad \bar{\partial} f = \partial_{\bar{z}} f d\bar{z}.$$

with $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$. Similarly, one can define the ∂ and $\bar{\partial}$ operators from $\mathbb{C}\Lambda^1$ to $\mathbb{C}\Lambda^2$ by setting

$$\partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{0,1}, \quad \bar{\partial}(\omega_{1,0} + \omega_{0,1}) := d\omega_{1,0}$$

if $\omega_{0,1} \in T_{0,1}^*M$ and $\omega_{1,0} \in T_{1,0}^*M$. In coordinates this is simply

$$\partial(udz + vd\bar{z}) = \partial v \wedge d\bar{z}, \quad \bar{\partial}(udz + vd\bar{z}) = \bar{\partial}u \wedge dz.$$

There is a natural operator, the Laplacian acting on functions and defined by

$$\Delta f := -2i \star \bar{\partial}\partial f = d^*d$$

where d^* is the adjoint of d through the metric g and \star is the Hodge star operator mapping Λ^2 to Λ^0 and induced by g as well.

2.2. Maslov Index and Boundary value problem for the $\bar{\partial}$ Operator. In this subsection we consider the setting where M is an oriented Riemann surface with boundary ∂M_0 and $\Gamma \subset \partial M_0$ is an open subset and we let $\Gamma_0 = \partial M_0 \setminus \Gamma$ be its complement in ∂M_0 . Following [22], we adopt the following notations: let $E \rightarrow M_0$ be a complex line bundle with complex structure $J : E \rightarrow E$ and let $D : C^\infty(M_0, E) \rightarrow C^\infty(M_0, T_{0,1}^* \otimes E)$ be a Cauchy-Riemann operator with smooth coefficients on M_0 , acting on sections of the bundle E . Observe that in the case when $E = M_0 \times \mathbb{C}$ is the trivial line bundle with the natural complex structure on M_0 , then D can be taken as the operator $\bar{\partial}$ introduced in (2). For $q > 1$, we define

$$D_F : W_F^{\ell,q}(M_0, E) \rightarrow W^{\ell-1,q}(M_0, T_{0,1}^*M_0 \otimes E)$$

where $F \subset E|_{\partial M_0}$ is a totally real subbundle (i.e. a subbundle such that $JF \cap F$ is the zero section) and D_F is the restriction of D to the L^q -based Sobolev space with ℓ derivatives and boundary condition F

$$W_F^{\ell,q}(M_0, E) := \{\xi \in W^{\ell,q}(M_0, E) \mid \xi(\partial M_0) \subset F\}.$$

In this setting, we have the following boundary value Riemann-Roch theorem stated in [22]:

Theorem 2.1. *Let $E \rightarrow M_0$ be a complex line bundle over an oriented compact Riemann surface with boundary and $F \subset E|_{\partial M_0}$ be a totally real subbundle. Let D be a smooth Cauchy-Riemann operator on E acting on $W^{\ell,q}(M_0, E)$ for some $q > 1$ and $\ell \in \mathbb{N}$. Then*

1) *The following operators are Fredholm*

$$D_F : W_F^{\ell,q}(M_0, E) \rightarrow W^{\ell-1,q}(M_0, T_{0,1}^*M_0 \otimes E)$$

$$D_F^* : W_F^{\ell,q}(M_0, T_{0,1}^*M_0 \otimes E) \rightarrow W^{\ell-1,q}(M_0, E).$$

2) *The real Fredholm index of D_F is given by*

$$\text{Ind}(D_F) = \chi(M_0) + \mu(E, F)$$

where $\chi(M_0)$ is the Euler characteristic of M_0 and $\mu(E, F)$ is the boundary Maslov index of the subbundle F .

3) *If $\mu(E, F) < 0$, then D_F is injective, while if $\mu(E, F) + 2\chi(M_0) > 0$ the operator D_F is surjective.*

In the case of a trivial bundle $E = M_0 \times \mathbb{C}$ and $\partial M_0 = \sqcup_{j=1}^m \partial_i M_0$ is a disjoint union of m circles, the Maslov index can be defined (see [22, p.554-555]) to be the degree of the map $\rho \circ \Lambda : \partial M_0 \rightarrow \partial M_0$ where

$$\Lambda|_{\partial_i M_0} : S^1 \simeq \partial_i M_0 \rightarrow \mathrm{GL}(1, \mathbb{C})/\mathrm{GL}(1, \mathbb{R})$$

is the natural map given by the totally real subbundle (the space $\mathrm{GL}(1, \mathbb{C})/\mathrm{GL}(1, \mathbb{R})$ being the space of totally real subbundles of \mathbb{C} over a circle) and

$$\rho : \mathrm{GL}(1, \mathbb{C})/\mathrm{GL}(1, \mathbb{R}) \rightarrow S^1, \quad \rho(A \cdot \mathrm{GL}(1, \mathbb{R})) := \det(A^2)/\det(A^* A).$$

As an application, we obtain the following (here and in what follows, $H^m(M_0) := W^{m,2}(M_0)$):

Corollary 2.2. (i) For $q > 1$ and $k \in \mathbb{N}_0$, let $\omega \in W^{k,q}(M_0, T_{0,1}^* M_0)$, then there exists $u \in W^{k+1,q}(M_0)$ holomorphic on M_0 , real valued on Γ_0 , such that $\bar{\partial}u = \omega$.
(ii) For $m > 1/2$, let $f \in H^m(\partial M_0)$ be a real valued function, then there exists a holomorphic function $v \in H^{m+\frac{1}{2}}(M_0)$ such that $\mathrm{Re}(v)|_{\Gamma_0} = f$.
(iii) For $k \in \mathbb{N}$ and $q > 1$, the space of $W^{k,q}(M_0)$ holomorphic functions on M_0 which are real valued on Γ_0 is infinite dimensional.

Proof. (i) Let $L \in \mathbb{N}$ be arbitrary large and let us identify the boundary as a disjoint union of circles $\partial M_0 = \coprod_{i=1}^m \partial_i M_0$ where each $\partial_i M_0 \simeq S^1$. Since Γ will be the piece of the boundary where we know the Cauchy data space, it is sufficient to assume that Γ is a connected non-empty open segment of $\partial_1 M_0 = S^1$, and which can thus be defined in a coordinate θ (respecting the orientation of the boundary) by $\Gamma = \{\theta \in S^1 \mid 0 < \theta < 2\pi/k\}$ for some integer k . Define the totally real subbundle of $F \subset E|_{\partial M_0} = \coprod_{j=1}^l (\partial_j M_0 \times \mathbb{C})$ by the following: on $\partial_1 M_0 \simeq S^1$ parametrized by $\theta \in [0, 2\pi]$, define $F_\theta = e^{ia(\theta)}\mathbb{R}$. Where $a : [0, 2\pi] \rightarrow \mathbb{R}$ is a smooth nondecreasing function such that $a(\theta) = 0$ in a neighbourhood $[0, \epsilon]$ of 0, $a(2\pi/k) = 2L\pi$ for some $L \in \mathbb{N}$, and $a(\theta) = 2L\pi$ for all $\theta > 2\pi/k$. For the rest of $\partial_2 M_0, \dots, \partial_m M_0$, we just let $F|_{\partial_i M_0} = S^1 \times \mathbb{R}$. The Maslov index $\mu(E, F)$ is then given by $2L$ and so, by theorem 2.1, D_F is surjective if $2\chi(M_0) + 2L > 0$. Since L can be taken as large as we want this achieves the proof of (i).

(ii) Let $w \in H^{m+\frac{1}{2}}(M_0)$ be a real function with boundary value f on ∂M_0 , then by (i) there exists $R \in H^{m+1/2}(M_0)$ such that $i\bar{\partial}R = -\bar{\partial}w$ and R purely real on Γ_0 , thus $v := iR + w$ is holomorphic such that $\mathrm{Re}(v) = f$ on Γ_0 .

(iii) Taking the subbundle F as in the proof of (i), we have that $\dim \ker D_F = \chi(M_0) + 2L$ if L satisfies $2\chi(M_0) + 2L > 0$, and since L can be taken as large as we like-, this concludes the proof. \square

Lemma 2.3. Let $\{p_0, p_1, \dots, p_n\} \subset M_0$ be a set of $n+1$ disjoint points. Let $c_1, \dots, c_K \in \mathbb{C}$, $N \in \mathbb{N}$, and let z be a complex coordinate near p_0 such that $p_0 = \{z = 0\}$. Then if $p_0 \in \mathrm{int}(M_0)$, there exists a holomorphic function f on M_0 with zeros of order at least N at each p_j , such that f is real on Γ_0 and $f(z) = c_0 + c_1 z + \dots + c_K z^K + O(|z|^{K+1})$ in the coordinate z . If $p_0 \in \partial M_0$, the same is true except that f is not necessarily real on Γ_0 .

Proof. First, using linear combinations and induction on K , it suffices to prove the Lemma for any K and $c_0 = \dots = c_{K-1} = 0$, which we now show. Consider the subbundle F as in the proof of (i) in Corollary 2.2. The Maslov index $\mu(E, F)$ is given by $2L$ and so for each $N \in \mathbb{N}$,

one can take L large enough to have $\mu(F, E) + 2\chi(M_0) \geq 2N(1 + n)$. Therefore by Theorem 2.1 the dimension of the kernel of $\bar{\partial}_F$ will be greater than $2(n + 1)N$. Now, since for each p_j and complex coordinate z_j near p_j , the map $u \rightarrow (u(p_j), \partial_{z_j} u(p_j), \dots, \partial_{z_j}^{N-1} u(p_j)) \in \mathbb{C}^N$ is linear, this implies that there exists a non-zero element $u \in \ker D_F$ which has zeros of order at least N at all p_j .

First, assume that $p_0 \in \text{int}(M_0)$ and we want the desired Taylor expansion at p_0 in the coordinate z . In the coordinate z , one has $u(z) = \alpha z^M + O(|z|^{M+1})$ for some $\alpha \neq 0$ and $M \geq N$. Define the function $r_K(z) = \chi(z) \frac{c_K}{\alpha} z^{-M+K}$ where $\chi(z)$ is a smooth cut-off function supported near p_0 and which is 1 near $p_0 = \{z = 0\}$. Since $M \geq N > 1$, this function has a pole at p_0 and trivially extends smoothly to $M_0 \setminus \{p_0\}$, which we still call r_K . Observe that the function is holomorphic in a neighbourhood of p_0 but not at p_0 where it is only meromorphic, so that in $M_0 \setminus \{p_0\}$, $\bar{\partial} r_K$ is a smooth and compactly supported section of $T_{0,1}^* M_0$ and therefore trivially extends smoothly to M_0 (by setting its value to be 0 at p_0) to a one form denoted ω_K . By the surjectivity assertion in Corollary 2.2, there exists a smooth function R_K satisfying $\bar{\partial} R_K = -\omega_K$ and that $R_K|_{\Gamma_0} \in \mathbb{R}$. We now have that $R_K + r_K$ is a holomorphic function on $M \setminus \{p_0\}$ meromorphic with a pole of order $M - K$ at p_0 , and in coordinate z one has $z^{M-K}(R_K(z) + r_K(z)) = c_K + O(|z|)$. Setting $f_K = u(R_K + r_K)$, we have the desired holomorphic function. Note that f also vanish to order N at all p_1, \dots, p_n since u does. This achieves the proof.

Now, if $p_0 \in \partial M_0$, we can consider a slightly larger smooth domain of M containing M_0 and we apply the the result above. \square

2.3. Morse holomorphic functions with prescribed critical points. The main result of this section is the following

Proposition 2.1. *Let p be an interior point of M_0 and $\epsilon > 0$ small. Then there exists a holomorphic function Φ on M_0 which is Morse on M_0 (up to the boundary) and real valued on Γ_0 , which has a critical point p' at distance less than ϵ from p and such that $\text{Im}(\Phi(p')) \neq 0$.*

Let \mathcal{O} be a connected open set of M such that $\bar{\mathcal{O}}$ is a smooth surface with boundary, with $M_0 \subset \bar{\mathcal{O}}$ and $\Gamma_0 \subset \partial \bar{\mathcal{O}}$. Fix $k > 2$ a large integer, we denote by $C^k(\bar{\mathcal{O}})$ the Banach space of C^k real valued functions on $\bar{\mathcal{O}}$. Then the set of harmonic functions on $\bar{\mathcal{O}}$ which are in the Banach space $C^k(\bar{\mathcal{O}})$ (and smooth in \mathcal{O} by elliptic regularity) is the kernel of the continuous map $\Delta : C^k(\bar{\mathcal{O}}) \rightarrow C^{k-2}(\bar{\mathcal{O}})$, and so it is a Banach subspace of $C^k(\bar{\mathcal{O}})$. The set $\mathcal{H} \subset C^k(\bar{\mathcal{O}})$ of harmonic functions u in $C^k(\bar{\mathcal{O}})$ such there exists $v \in C^k(\bar{\mathcal{O}})$ harmonic with $u + iv$ holomorphic on \mathcal{O} is a Banach subspace of $C^k(\bar{\mathcal{O}})$ of finite codimension. Indeed, let $\{\gamma_1, \dots, \gamma_N\}$ be a homology basis for \mathcal{O} , then

$$\mathcal{H} = \ker L, \text{ with } L : \ker \Delta \cap C^k(\bar{\mathcal{O}}) \rightarrow \mathbb{C}^N \text{ defined by } L(u) := \left(\frac{1}{\pi i} \int_{\gamma_j} \partial u \right)_{j=1, \dots, N}.$$

For all $\Gamma'_0 \subset \partial M_0$ such that the complement of Γ'_0 contains an open subset, we define

$$\mathcal{H}_{\Gamma'_0} := \{u \in \mathcal{H}; u|_{\Gamma'_0} = 0\}.$$

We now show

Lemma 2.4. *The set of functions $u \in \mathcal{H}_{\Gamma'_0}$ which are Morse in \mathcal{O} is residual (i.e. a countable intersection of open dense sets) in $\mathcal{H}_{\Gamma'_0}$ with respect to the $C^k(\bar{\mathcal{O}})$ topology.*

Proof. We use an argument very similar to those used by Uhlenbeck [30]. We start by defining $m : \mathcal{O} \times \mathcal{H}_{\Gamma'_0} \rightarrow T^*\mathcal{O}$ by $(p, u) \mapsto (p, du(p)) \in T_p^*\mathcal{O}$. This is clearly a smooth map, linear in the second variable, moreover $m_u := m(\cdot, u) = (\cdot, du(\cdot))$ is Fredholm since \mathcal{O} is finite dimensional. The map u is a Morse function if and only if m_u is transverse to the zero section, denoted $T_0^*\mathcal{O}$, of $T^*\mathcal{O}$, ie. if

$$\text{Image}(D_p m_u) + T_{m_u(p)}(T_0^*\mathcal{O}) = T_{m_u(p)}(T^*\mathcal{O}), \quad \forall p \in \mathcal{O} \text{ such that } m_u(p) = (p, 0),$$

which is equivalent to the fact that the Hessian of u at critical points is non-degenerate (see for instance Lemma 2.8 of [30]). We recall the following transversality theorem ([30, Th.2]):

Theorem 2.5. *Let $m : X \times \mathcal{H}_{\Gamma'_0} \rightarrow W$ be a C^k map, where X , $\mathcal{H}_{\Gamma'_0}$, and W are separable Banach manifolds with W and X of finite dimension. Let $W' \subset W$ be a submanifold such that $k > \max(1, \dim X - \dim W + \dim W')$. If m is transverse to W' then the set $\{u \in \mathcal{H}_{\Gamma'_0}; m_u \text{ is transverse to } W'\}$ is dense in $\mathcal{H}_{\Gamma'_0}$, more precisely it is a residual set.*

We want to apply it with $X := \mathcal{O}$, $W := T^*\mathcal{O}$ and $W' := T_0^*\mathcal{O}$, and the map m is defined above. We have thus proved Lemma 2.4 if one can show that m is transverse to W' . Let (p, u) such that $m(p, u) = (p, 0) \in W'$. Then identifying $T_{(p,0)}(T^*\mathcal{O})$ with $T_p\mathcal{O} \oplus T_p^*\mathcal{O}$, one has

$$D_{(p,u)}m(z, v) = (z, dv(p) + \text{Hess}_p(u)z)$$

where $\text{Hess}_p u$ is the Hessian of u at the point p , viewed as a linear map from $T_p\mathcal{O}$ to $T_p^*\mathcal{O}$. To prove that m is transverse to W' we need to show that $(z, v) \rightarrow (z, dv(p) + \text{Hess}_p(u)z)$ is onto from $T_p\mathcal{O} \oplus \mathcal{H}_{\Gamma'_0}$ to $T_p\mathcal{O} \oplus T_p^*\mathcal{O}$, which is realized for instance if the map $v \rightarrow dv(p)$ from $\mathcal{H}_{\Gamma'_0}$ to $T_p^*\mathcal{O}$ is onto. But from Lemma 2.3, we know that there exist holomorphic functions v and \tilde{v} on \mathcal{O} such that v and \tilde{v} are purely real on Γ'_0 . Clearly the imaginary parts of v and \tilde{v} belong to $\mathcal{H}_{\Gamma'_0}$. Furthermore, for a given complex coordinate z near $p = \{z = 0\}$, we can arrange them to have series expansion $v(z) = z + O(|z|^2)$ and $\tilde{v}(z) = iz + O(|z|^2)$ around the point p . We see, by coordinate computation of the exterior derivative of $\text{Im}(v)$ and $\text{Im}(\tilde{v})$, that $d\text{Im}(v)(p)$ and $d\text{Im}(\tilde{v})(p)$ are linearly independent at the point p . This shows our claim and ends the proof of Lemma 2.4 by using Theorem 2.5. \square

We now proceed to show that the set of all functions $u \in \mathcal{H}_{\Gamma'_0}$ such that u has no degenerate critical points on Γ'_0 is also residual.

Lemma 2.6. *For all $p \in \Gamma'_0$ and $k \in \mathbb{N}$, there exists a holomorphic function $u \in C^k(\bar{\mathcal{O}})$, such that $\text{Im}(u)|_{\Gamma'_0} = 0$ and $\partial u(p) \neq 0$.*

Proof. The proof is quite similar to that of Lemma 2.3. By Lemma 2.3, we can choose a holomorphic function $v \in C^k(\bar{\mathcal{O}})$ such that $v(p) = 0$ and $\text{Im}(v)|_{\Gamma'_0} = 0$, then either $\partial v(p) \neq 0$ and we are done, or $\partial v(p) = 0$. Assume now the second case and let $M \in \mathbb{N}$ be the order of p as a zero of v . By Riemann mapping theorem, there is a conformal mapping from a neighbourhood U_p of p in \mathcal{O} to a neighbourhood $\{|z| < \epsilon, \text{Im}(z) \geq 0\}$ of the real line $\text{Im}(z) = 0$ in \mathbb{C} , and one can assume that $p = \{z = 0\}$ in these complex coordinates. Take $r(z) = \chi(z)z^{-M+1}$ where $\chi \in C_0^\infty(|z| \leq \epsilon)$ is a real valued function with $\chi(z) = 1$ in $\{|z| < \epsilon/2\}$. Then $\bar{\partial}r$ vanishes in the pointed disc $0 < |z| < \epsilon/2$ and it is a compactly supported smooth section of $T_{1,0}^*\bar{\mathcal{O}}$ outside, it can thus be extended trivially to a smooth section of $T_{1,0}^*\bar{\mathcal{O}}$ denoted by ω . We can then use (i) of Corollary 2.2: there is a function R such that $\bar{\partial}R = -\omega$ and $\text{Im}(R)|_{\Gamma'_0} = 0$, and so $\bar{\partial}(R+r) = 0$ in $\mathcal{O} \setminus \{p\}$ and $R+r$ is real valued on Γ'_0 (remark that r is real valued on Γ'_0) and has a pole at p of order exactly $M-1$. We

conclude that $u := v(R+r)$ satisfies the desired properties, it vanishes at p but with non zero complex derivative at p . \square

Lemma 2.7. *Let $\Gamma'_0 \subset \partial\mathcal{O}$ be an open set of the boundary. Let $\phi : \mathcal{O} \rightarrow \mathbb{R}$ be a harmonic function with $\phi|_{\Gamma'_0} = 0$. Let $p \in \Gamma'_0$ be a critical point of ϕ , then it is nondegenerate if and only if $\partial_\tau \partial_\nu u \neq 0$ where ∂_τ and ∂_ν denote respectively the tangential and normal derivatives along the boundary.*

Proof. By Riemman mapping theorem, there is a conformal transformation mapping a neighbourhood of p in $\bar{\mathcal{O}}$ to a half-disc $D := \{|z| < \epsilon, \text{Im}(z) \geq 0\}$ and $\partial\bar{\mathcal{O}} = \{\text{Im}(z) = 0\}$ near p . Denoting $z = x + iy$, one has $(\partial_x^2 + \partial_y^2)\phi = 0$ in D and $\partial_x^2 \phi|_{y=0} = 0$, which implies $\partial_y^2 \phi(p) = 0$. Since $\partial_\nu = e^f \partial_y$ and $\partial_\tau = e^f \partial_x$ for some smooth function f , and since $d\phi(p) = 0$, the conclusion is then straightforward. \square

Let $N^* \partial\bar{\mathcal{O}}$ be the conormal-bundle of $\partial\bar{\mathcal{O}}$ and $N^* \Gamma'_0$ be the restriction of this bundle to Γ'_0 . Denote the zero sections of these bundles respectively by $N_0^* \partial\bar{\mathcal{O}}$ and $N_0^* \Gamma'_0$. We now define the map

$$b : \Gamma'_0 \times \mathcal{H}_{\Gamma'_0} \rightarrow N^* \Gamma'_0, \quad b(p, u) := (p, \partial_\nu u).$$

For a fixed $u \in \mathcal{H}_{\Gamma'_0}$, we also define $b_u(\cdot) := b(\cdot, u)$. Simple computations yield the

Lemma 2.8. *Suppose that $p \in \Gamma'_0$ is such that $\partial_\nu u(p) = 0$, then $\partial_\tau \partial_\nu u(p) \neq 0$ if and only if*

$$\text{Image}(D_p b_u) + T_{(p,0)}(N_0^* \Gamma'_0) = T_{(p,0)}(N^* \Gamma'_0).$$

Proof. This can be seen by the fact that for all $p \in \Gamma'_0$ such that $b_u(p) = (p, 0)$,

$$D_p b_u : T_p \Gamma'_0 \rightarrow T_{(p,0)}(N^* \Gamma'_0) \simeq T_p \Gamma'_0 \oplus N_p^* \Gamma'_0$$

is given by $w \mapsto (w, \partial_\tau \partial_\nu u(p)w)$. \square

At a point (p, u) such that $b(p, u) = 0$, a simple computation yields that the differential $D_{(p,u)} b : T_p \Gamma'_0 \times \mathcal{H}_{\Gamma'_0} \rightarrow T_{(p, \partial_\nu u(p))}(N^* \Gamma'_0)$ is given by $(w, u') \mapsto (w, \partial_\tau \partial_\nu u(p)w + \partial_\nu u'(p))$. This observation combined with Lemma 2.6 shows that for all $(p, u) \in \Gamma'_0 \times \mathcal{H}_{\Gamma'_0}$ such that $b(p, u) = (p, 0)$, b is transverse to $N_0^* \Gamma'_0$ at $(p, 0)$. Now we can apply Theorem 2.5 with $X = \Gamma'_0$, $W = N^* \Gamma'_0$ and $W' = N_0^* \Gamma'_0$ we see that the set $\{u \in \mathcal{H}_{\Gamma'_0}; b_u \text{ is transverse to } N_0^* \Gamma'_0\}$ is residual in $\mathcal{H}_{\Gamma'_0}$. In view of Lemmas 2.7, we deduce the

Lemma 2.9. *The set of functions $u \in \mathcal{H}_{\Gamma'_0}$ such that u has no degenerate critical point on Γ'_0 is residual in $\mathcal{H}_{\Gamma'_0}$.*

Observing the general fact that finite intersection of residual sets remains residual, the combination of Lemma 2.9 and Lemma 2.4 yields

Corollary 2.10. *The set of functions $u \in \mathcal{H}_{\Gamma'_0}$ which are Morse in \mathcal{O} and have no degenerate critical points on Γ'_0 is residual in $\mathcal{H}_{\Gamma'_0}$ with respect to the $C^k(\bar{\mathcal{O}})$ topology. In particular, it is dense.*

We are now in a position to give a proof of the main proposition of this section.

Proof of Proposition 2.1. As explained above, choose \mathcal{O} in such a way that $\bar{\mathcal{O}}$ is a smooth surface with boundary, containing M_0 , that $\Gamma_0 \subset \partial\mathcal{O}$ and \mathcal{O} contains $\partial M_0 \setminus \bar{\Gamma}_0$. Let Γ'_0 be

an open subset of the boundary of $\bar{\mathcal{O}}$ such that the closure of Γ_0 is contained in Γ'_0 and $\partial\bar{\mathcal{O}} \setminus \bar{\Gamma'_0} \neq \emptyset$. Let p be an interior point of M_0 . By lemma 2.3, there exists a holomorphic function $f = u + iv$ on $\bar{\mathcal{O}}$ such that f is purely real on Γ'_0 , $v(p) = 1$, and $df(p) = 0$ (thus $v \in \mathcal{H}_{\Gamma'_0}$).

By Corollary 2.10, there exist a sequence $(v_j)_j$ of Morse functions $v_j \in \mathcal{H}_{\Gamma'_0}$ such that $v_j \rightarrow v$ in $C^k(M_0)$ for any fixed k large. By Cauchy integral formula, there exist harmonic conjugates u_j of v_j such that $f_j := u_j + iv_j \rightarrow f$ in $C^k(M_0)$. Let $\epsilon > 0$ be small and let $U \subset \mathcal{O}$ be a neighbourhood containing p and no other critical points of f , and with boundary a smooth circle of radius ϵ . In complex local coordinates near p , we can identify ∂f and ∂f_j to holomorphic functions on an open set of \mathbb{C} . Then by Rouché's theorem, it is clear that ∂f_j has precisely one zero in U and v_j never vanishes in U if j is large enough.

Fix Φ to be one of the f_j for j large enough. By construction, Φ is Morse in \mathcal{O} and has no degenerate critical points on $\bar{\Gamma_0} \subset \Gamma'_0$. We notice that, since the imaginary part of Φ vanishes on all of Γ'_0 , it is clear from the reflection principle applied after using the Riemann mapping theorem (as in the proof of Lemma 2.7) that no point on $\bar{\Gamma_0} \subset \Gamma'_0$ can be an accumulation point for critical points. Now $\partial M_0 \setminus \bar{\Gamma_0}$ is contained in the interior of \mathcal{O} and therefore no points on $\partial M_0 \setminus \bar{\Gamma_0}$ can be an accumulation point of critical points. Since Φ is Morse in the interior of \mathcal{O} , there are no degenerate critical points on $\partial M_0 \setminus \bar{\Gamma_0}$. This ends the proof. \square

3. CARLEMAN ESTIMATE FOR HARMONIC WEIGHTS WITH CRITICAL POINTS

In this section, we prove a Carleman estimate using harmonic weight with non-degenerate critical points, in way similar to [16]. Let us define $\Gamma_0 := \{p \in \partial M_0; \partial_\nu \varphi(p) = 0\}$ and let $\Gamma := \partial M_0 \setminus \Gamma_0$ its complement.

Proposition 3.1. *Let (M_0, g) be a smooth Riemann surface with boundary, and let $\varphi : M_0 \rightarrow \mathbb{R}$ be a $C^k(M_0)$ harmonic Morse function for k large. Then for all $V \in L^\infty(M_0)$ there exists an $h_0 > 0$ such that for all $h \in (0, h_0)$ and $u \in C^\infty(M)$ with $u|_{\partial M_0} = 0$, we have*

$$(3) \quad \begin{aligned} & \frac{1}{h} \|u\|_{L^2(M_0)}^2 + \frac{1}{h^2} \|u|d\varphi|\|_{L^2(M_0)}^2 + \|du\|_{L^2(M_0)}^2 + \|\partial_\nu u\|_{L^2(\Gamma_0)}^2 \\ & \leq C \left(\|e^{-\varphi/h}(\Delta_g + V)e^{\varphi/h}u\|_{L^2(M_0)}^2 + \frac{1}{h} \|\partial_\nu u\|_{L^2(\Gamma)}^2 \right) \end{aligned}$$

where ∂_ν is the exterior unit normal vector field to ∂M_0 .

Proof. We start by modifying the weight as follows: if $\varphi_0 := \varphi : M_0 \rightarrow \mathbb{R}$ is a real valued harmonic Morse function with critical points $\{p_1, \dots, p_N\}$ in the interior of M_0 , we let $\varphi_j : M_0 \rightarrow \mathbb{R}$ be harmonic functions such that p_j is not a critical point of φ_j for $j = 1, \dots, N$, their existence is insured by Lemma 2.3. For all $\epsilon > 0$, we define the convexified weight $\varphi_\epsilon := \varphi - \frac{h}{2\epsilon} (\sum_{j=0}^N |\varphi_j|^2)$.

To prove the estimate, we shall localize in charts Ω_j covering the surfaces. These charts will be taken so that if $\Omega_j \cap \partial M_0 \neq \emptyset$, then $\Omega_j \cap \partial M_0 \simeq S^1$ is a connected component of ∂M_0 . Moreover, by Riemann mapping theorem (e.g. Lemma 3.2 of [23]), this chart can be taken to be a neighbourhood of $|z| = 1$ in $\{z \in \mathbb{C}; |z| \leq 1\}$ and such that the metric g is conformal to the Euclidean metric $|dz|^2$.

Lemma 3.1. *Let Ω be a chart of M_0 as above and $\varphi_\epsilon : \Omega \rightarrow \mathbb{R}$ be as above. Then there are constants $C, C' > 0$ such that for all $\omega \in C^\infty(M)$ supported in Ω and $h > 0$ small enough, the following estimate holds:*

$$(4) \quad \frac{C}{\epsilon} \|\omega\|_{L^2(M_0)}^2 + C' \left(-\operatorname{Im}(\langle \partial_\tau \omega, \omega \rangle_{L^2(\partial M_0)}) + \frac{1}{h} \int_{\partial M_0} |\omega|^2 \partial_\nu \varphi_\epsilon \, dv_g \right) \leq \|e^{-\varphi_\epsilon/h} \bar{\partial} e^{\varphi_\epsilon/h} \omega\|_{L^2(M_0)}^2$$

where ∂_ν and ∂_τ denote respectively the exterior pointing normal vector fields and its rotation by an angle $+\pi/2$.

Proof. We use complex coordinates $z = x + iy$ in the chart Ω where u is supported. Observe that the Lebesgue measure $dx dy$ is bounded below and above by dv_g , g is conformal to $|dz|^2$ and the boundary terms in (4) depend only on the conformal class, it suffices to prove the estimates with respect to $dx dy$ and the Euclidean metric. We thus integrate by parts with respect to $dx dy$

$$(5) \quad \begin{aligned} 4\|e^{-\varphi_\epsilon/h} \bar{\partial} e^{\varphi_\epsilon/h} u\|^2 &= \|(\partial_x + \frac{i\partial_y \varphi_\epsilon}{h})u + (i\partial_y + \frac{\partial_x \varphi_\epsilon}{h})u\|^2 \\ &= \|(\partial_x + \frac{i\partial_y \varphi_\epsilon}{h})u\|^2 + \|(i\partial_y + \frac{\partial_x \varphi_\epsilon}{h})u\|^2 \\ &\quad + \frac{2}{h} \int_\Omega \left(\Delta \varphi_\epsilon |u|^2 - \frac{1}{2} \partial_x \varphi_\epsilon \cdot \partial_x |u|^2 - \frac{1}{2} \partial_y \varphi_\epsilon \cdot \partial_y |u|^2 \right) \\ &\quad + \frac{2}{h} \int_{\partial M_0} \partial_\nu \varphi_\epsilon |u|^2 - 2 \int_{M_0} \left(\partial_x \operatorname{Re}(u) \cdot \partial_y \operatorname{Im}(u) - \partial_x \operatorname{Im}(u) \cdot \partial_y \operatorname{Re}(u) \right) \\ &= \|(\partial_x + \frac{i\partial_y \varphi_\epsilon}{h})u\|^2 + \|(i\partial_y + \frac{\partial_x \varphi_\epsilon}{h})u\|^2 + \frac{1}{h} \int_\Omega \Delta \varphi_\epsilon |u|^2 \\ &\quad + \frac{1}{h} \int_{\partial M_0} \partial_\nu \varphi_\epsilon |u|^2 + 2 \int_{\partial M_0} \partial_\tau \operatorname{Re}(u) \cdot \operatorname{Im}(u). \end{aligned}$$

where $\Delta := -(\partial_x^2 + \partial_y^2)$, ∂_ν is the exterior pointing normal vector field to the boundary and ∂_τ is the tangent vector field to the boundary (i.e. ∂_ν rotated with an angle $\pi/2$) for the Euclidean metric $|dz|^2$. Then $\langle u \Delta \varphi_\epsilon, u \rangle = \frac{h}{\epsilon} (|d\varphi_0|^2 + |d\varphi_1|^2 + \dots + |d\varphi_N|^2) |u|^2$, since φ_j are harmonic, so the proof follows from the fact that $|d\varphi_0|^2 + |d\varphi_1|^2 + \dots + |d\varphi_N|^2$ is uniformly bounded away from zero. \square

The main step to go from (4) to (3) is the following lemma which is a slight modification of the proof in [16]:

Lemma 3.2. *With the same assumptions as in Proposition 3.1, and if Ω is either an interior chart of (M, g) or a chart containing a whole boundary connected component, then there are positive constants c and C such that for all $\epsilon > 0$ small, all $0 < h \ll \epsilon$ and all smooth function u supported in Ω with $u|_{\partial M_0} = 0$, we have*

$$(6) \quad \begin{aligned} &C \left(\|e^{-\varphi_\epsilon/h} \Delta_g e^{\varphi_\epsilon/h} u\|_{L^2(M_0)}^2 + \frac{1}{h} \|\partial_\nu u\|_{L^2(\Gamma)}^2 \right) \geq \\ &\frac{c}{\epsilon} \left(\frac{1}{h} \|u\|_{L^2(M)}^2 + \frac{1}{h^2} \|u|d\varphi|\|_{L^2(M)}^2 + \frac{1}{h^2} \|u|d\varphi_\epsilon|\|_{L^2(M)}^2 + \|du\|_{L^2(M)}^2 \right) + \|\partial_\nu u\|_{L^2(\partial M_0)}^2 \end{aligned}$$

where ∂_ν denote the unit normal vector field to ∂M_0 .

Proof. Since the norms induced by the metric g in the chart are conformal to Euclidean norms, and there is f smooth such that $\Delta_g = -e^{2f}(\partial_x^2 + \partial_y^2) = e^{2f}\Delta$ in the complex coordinate chart, it suffices to get the estimate (6) for Euclidean norms and Laplacian. Clearly, we can assume $u \in H_0^1(M)$ to be real valued without loss of generality. Now let $Q(z)$ be a holomorphic function in Ω which has no zeros, then by (5) we have

$$\|Qe^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 = 16\|e^{-\varphi_\epsilon/h}\bar{\partial}e^{\varphi_\epsilon/h}Qe^{-\varphi_\epsilon/h}\partial e^{\varphi_\epsilon/h}u\|^2 \geq \frac{C}{\epsilon}\|\partial u + \frac{\partial\varphi_\epsilon}{h}u\|^2 - 4\text{Im}\left(\int_{\partial M_0}\partial_\tau\omega.\bar{\omega}\right) + \frac{4}{h}\int_{\partial M_0}|\omega|^2\partial_\nu\varphi_\epsilon$$

with $\omega := Qe^{-\varphi_\epsilon/h}\partial e^{\varphi_\epsilon/h}u$ and here everything is measured with respect to Euclidean metric $dx^2 + dy^2$ and measure $dx dy$. Since $u|_{\partial M_0} = 0$, one has $\omega|_{\partial M_0} = (A + iB)\partial_\nu u$ where $A + iB = Q(\langle\partial_\nu, \partial_x\rangle - i\langle\partial_\nu, \partial_y\rangle)$ and $-\text{Im}(\partial_\tau\omega.\bar{\omega}) = (\partial_\tau A.B - A\partial_\tau B)|\partial_\nu u|^2$ we deduce that for some $c > 0$

$$(7) \quad \begin{aligned} & \|Qe^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 \geq \\ & \frac{c}{\epsilon}\left(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \frac{2}{h}\langle\partial_x u, u\partial_x\varphi_\epsilon\rangle + \frac{2}{h}\langle\partial_y u, u\partial_y\varphi_\epsilon\rangle\right) \\ & + 4\int_{\partial M_0}(A\partial_\tau B - \partial_\tau AB)|\partial_\nu u|^2 + \frac{4}{h}\int_{\partial M_0}|Q|^2|\partial_\nu u|^2\partial_\nu\varphi_\epsilon. \end{aligned}$$

Using the fact that u is real valued, that φ is harmonic and that $\sum_{j=0}^N|d\varphi_j|^2$ is uniformly bounded away from 0, we see that

$$\frac{2}{h}\langle\partial_x u, u\partial_x\varphi_\epsilon\rangle + \frac{2}{h}\langle\partial_y u, u\partial_y\varphi_\epsilon\rangle = \frac{1}{h}\langle u, u\Delta\varphi_\epsilon\rangle \geq \frac{C}{\epsilon}\|u\|^2$$

for some $C > 0$ and therefore,

$$(8) \quad \|Qe^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 \geq \frac{c}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \frac{C}{\epsilon}\|u\|^2) + \text{boundary terms}.$$

Now if the diameter of the support of u is chosen small (with size depending only on $|\text{Hess}\varphi_0|(p)|$) with a unique critical point p of φ_0 inside, one can use integration by parts and the fact that the critical point is non-degenerate to obtain

$$(9) \quad \|\bar{\partial}u\|^2 + \frac{1}{h^2}\|u|d\varphi_0|\|^2 \geq \frac{1}{h}\left|\int\partial_{\bar{z}}(u^2)\overline{\partial_z\varphi_0}dx dy\right| \geq \frac{1}{h}\left|\int u^2\overline{\partial_z^2\varphi_0}dx dy\right| \geq \frac{C'}{h}\|u\|^2$$

for some $C' > 0$. Clearly the same estimate holds trivially if Ω does not contain critical point of φ_0 . Using a partition of unity $(\theta_j)_j$ in Ω and absorbing terms of the form $\|u\bar{\partial}\theta_j\|^2$ into the right hand side, one obtains (9) for any function u supported in Ω and vanishing at the boundary. Thus, combining with (8), there are positive constants c, c', C'' such that for h small enough

$$\begin{aligned} & \frac{c}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \frac{C}{\epsilon}\|u\|^2) \geq \frac{c}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_0|\|^2 - \frac{C''}{\epsilon^2}\|u\|^2) \\ & \geq \frac{c'}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_0|\|^2 + \frac{1}{h}\|u\|^2). \end{aligned}$$

Combining now with (8) and using that $|Q|$ is bounded below gives

$$\|e^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 \geq \frac{c'}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi|\|^2 + \frac{1}{h}\|u\|^2) + \text{boundary terms}.$$

Let us now discuss the boundary terms in (7). If φ_j are taken so that $\partial_\nu \varphi_j = 0$ on Γ_0 , then $\partial_\nu \varphi_\epsilon = 0$ on Γ_0 and $\partial_\nu \varphi_\epsilon = \partial_\nu \varphi + O(h/\epsilon)$ on Γ and thus

$$\frac{1}{h} \int_{\partial M_0} |Q|^2 |\partial_\nu u|^2 |\partial_\nu \varphi_\epsilon| \leq \frac{C_1}{h} \int_\Gamma |\partial_\nu u|^2$$

for some constant C_1 . We finally claim that there exist Q with no zeros in Ω such that $A\partial_\tau B - B\partial_\tau A$ is bounded below by a positive constant on $\partial M_0 \cap \Omega$. Indeed, since the chart near a connected component can be taken to be an interior neighbourhood of the circle $|z| = 1$ in \mathbb{C} , one can take $A + iB = e^{it}$ where $t \in S^1$ parametrizes the boundary component, so that $A\partial_\tau B - B\partial_\tau A = 1$ since $\partial_\tau = \partial_t$ for the Euclidean metric. Since moreover $\langle \partial_\nu, \partial_x \rangle - i \langle \partial_\nu, \partial_y \rangle = A - iB = e^{-it}$, we deduce that on the boundary $Q(t) = e^{2it}$ and so it suffices to take $Q(z) = z^2$. This achieves the proof. \square

Proof of Proposition 3.1. Using triangular inequality and absorbing the term $\|Vu\|^2$ into the left hand side of (3), it suffices to prove (3) with Δ_g instead of $\Delta_g + V$. Let $v \in C_0^\infty(M)$, we have by Lemma 3.2 that there exist constants $c, c', C, C' > 0$ such that

$$\begin{aligned} & \frac{c}{\epsilon} \left(\frac{1}{h} \|e^{-\varphi_\epsilon/h} v\|^2 + \frac{1}{h^2} \|e^{-\varphi_\epsilon/h} v |d\varphi|\|^2 + \frac{1}{h^2} \|e^{-\varphi_\epsilon/h} v |d\varphi_\epsilon|\|^2 + \|e^{-\varphi_\epsilon/h} dv\|^2 \right) + \|e^{-\varphi_\epsilon/h} \partial_\nu v\|_{\Gamma_0}^2 \\ & \leq \sum_j \frac{c'}{\epsilon} \left(\frac{1}{h} \|e^{-\varphi_\epsilon/h} \chi_j v\|^2 + \frac{1}{h^2} \|e^{-\varphi_\epsilon/h} \chi_j v |d\varphi|\|^2 + \frac{1}{h^2} \|e^{-\varphi_\epsilon/h} \chi_j v |d\varphi_\epsilon|\|^2 \right. \\ & \quad \left. + \|e^{-\varphi_\epsilon/h} d(\chi_j v)\|^2 \right) + \|e^{-\varphi_\epsilon/h} \partial_\nu v\|_{\Gamma_0}^2 \\ & \leq C \left(\sum_j \|e^{-\varphi_\epsilon/h} \Delta_g(\chi_j v)\|^2 + \|e^{-\varphi_\epsilon/h} \partial_\nu v\|_\Gamma^2 \right) \\ & \leq C' \left(\|e^{-\varphi_\epsilon/h} \Delta_g v\|^2 + \|e^{-\varphi_\epsilon/h} v\|^2 + \|e^{-\varphi_\epsilon/h} dv\|^2 + \|e^{-\varphi_\epsilon/h} \partial_\nu v\|_\Gamma^2 \right) \end{aligned}$$

where $(\chi_j)_j$ is a partition of unity associated to the complex charts Ω_j on M . Since constants on both sides are independent of ϵ and h , we can take ϵ small enough so that $C' \|e^{-\varphi_\epsilon/h} v\|^2 + C' \|e^{-\varphi_\epsilon/h} dv\|^2$ can be absorbed to the left side. Now set $v = e^{\varphi_\epsilon/h} w$ with $w|_{\partial M_0} = 0$, then we have

$$\begin{aligned} & \frac{1}{h} \|w\|^2 + \frac{1}{h^2} \|w |d\varphi|\|^2 + \frac{1}{h^2} \|w |d\varphi_\epsilon|\|^2 + \|dw\|^2 + \|\partial_\nu w\|_{\Gamma_0}^2 \\ & \leq C \left(\|e^{-\varphi_\epsilon/h} \Delta_g e^{\varphi_\epsilon/h} w\|^2 + \|\partial_\nu w\|_\Gamma^2 \right) \end{aligned}$$

Finally, fix $\epsilon > 0$ and set $u := e^{\frac{1}{\epsilon} \sum_{j=0}^N |\varphi_j|^2} w$ and use the fact that $e^{\frac{1}{\epsilon} \sum_{j=0}^N |\varphi_j|^2}$ is independent of h and bounded uniformly away from zero and above, we then obtain the desired estimate for $0 < h \ll \epsilon$. \square

4. COMPLEX GEOMETRIC OPTICS ON A RIEMANN SURFACE

As explained in the Introduction, the method for identifying the potential at a point p is to construct complex geometric optic solutions depending on a small parameter $h > 0$, with phase a Carleman weight (here a Morse holomorphic function), and such that the phase has a non-degenerate critical point at p , in order to apply the stationary phase method. In this section, the potential V has the regularity $V \in C^{1,\alpha}(M_0)$ for some $\alpha > 0$.

Choose $p \in \text{int}(M_0)$ such that there exists a holomorphic function $\Phi = \varphi + i\psi$ which is Morse on M_0 , C^k in M_0 for large $k \in \mathbb{N}$ and such that $\partial\Phi(p) = 0$ and Φ has only finitely many critical points in M_0 . Furthermore we ask that Φ is purely real on Γ_0 . By Proposition 2.1 such points p form a dense subset of M_0 . Given such a holomorphic function, the purpose of this section is to construct solutions u on M_0 of $(\Delta + V)u = 0$ of the form

$$(10) \quad u = e^{\Phi/h}(a + ha_0 + r_1) + \overline{e^{\Phi/h}(a + ha_0 + r_1)} + e^{\varphi/h}r_2 \quad \text{with} \quad u|_{\Gamma_0} = 0$$

for $h > 0$ small, where a is holomorphic and $u \in C^k(M_0)$ for large $k \in \mathbb{N}$, $a_0 \in H^2(M_0)$ is holomorphic, moreover $a(p) \neq 0$ and a vanishes to high order at all other critical points $p' \in M_0$ of Φ . Furthermore, we ask that the holomorphic function a is purely imaginary on Γ_0 . The existence of such a holomorphic function is a consequence of Lemma 2.3. Given such a holomorphic function on M_0 we consider a compactly supported extension to M , still denoted a .

The remainder terms r_1, r_2 will be controlled as $h \rightarrow 0$ and have particular properties near the critical points of Φ . More precisely, r_2 will be a $O_{L^2}(h^{3/2}|\log h|)$ and r_1 will be of the form $h\tilde{r}_{12} + o_{L^2}(h)$ where \tilde{r}_{12} is independent of h , which can be used to obtain sufficient informations from the stationary phase method in the identification process.

4.1. Construction of r_1 . We shall construct r_1 to satisfy

$$e^{-\Phi/h}(\Delta_g + V)e^{\Phi/h}(a + r_1) = O_{L^2}(h|\log h|)$$

and $r_1 = r_{11} + hr_{12}$. We let G be the Green operator of the Laplacian on the smooth surface with boundary M_0 with Dirichlet condition, so that $\Delta_g G = \text{Id}$ on $L^2(M_0)$. In particular this implies that $\partial\bar{\partial}G = \frac{i}{2}\star^{-1}$ where \star^{-1} is the inverse of \star mapping functions to 2-forms. We extend a to be a compactly supported C^k function on M_0 and we will search for $r_1 \in H^2(M_0)$ satisfying $\|r_1\|_{L^2} = O(h)$ and

$$(11) \quad e^{-2i\psi/h}\partial e^{2i\psi/h}r_1 = -\partial G(aV) + \omega + O_{H^1}(h|\log h|)$$

where ω is a smooth holomorphic 1-form on M_0 . Indeed, using the fact that Φ is holomorphic we have

$$e^{-\Phi/h}\Delta_g e^{\Phi/h} = -2i\star\bar{\partial}e^{-\Phi/h}\partial e^{\Phi/h} = -2i\star\bar{\partial}e^{-\frac{1}{h}(\Phi-\bar{\Phi})}\partial e^{\frac{1}{h}(\Phi-\bar{\Phi})} = -2i\star\bar{\partial}e^{-2i\psi/h}\partial e^{2i\psi/h}$$

and applying $-2i\star\bar{\partial}$ to (11), we obtain (note that $\partial G(aV) \in C^{2,\alpha}(M_0)$ by elliptic regularity)

$$e^{-\Phi/h}(\Delta_g + V)e^{\Phi/h}r_1 = -aV + O_{L^2}(h|\log h|).$$

We will choose ω to be a smooth holomorphic 1-form on M_0 such that at all critical point p' of Φ in M_0 , the form $b := \partial G(aV) - \omega$ with value in $T_{1,0}^*M_0$ vanish to the highest possible order. Writing $b = b(z)dz$ in local complex coordinates, $b(z)$ is $C^{2+\alpha}$ by elliptic regularity and we have $-2i\partial_{\bar{z}}b(z) = aV$, therefore $\partial_z\partial_{\bar{z}}b(p') = \partial_{\bar{z}}^2b(p') = 0$ at each critical point $p' \neq p$ by construction of the function a . Therefore, we deduce that at each critical point $p' \neq p$, $\partial G(aV)$ has Taylor series expansion $\sum_{j=0}^2 c_j z^j + O(|z|^{2+\alpha})$ for some N large. That is, all the lower order terms of the Taylor expansion of $\partial G(aV)$ around p' are polynomials of z only.

Lemma 4.1. *Let $\{p_0, \dots, p_N\}$ be finitely many points on M_0 and let θ be a $C^{2,\alpha}$ section of $T_{1,0}^*M_0$. Then there exists a C^k holomorphic function f on M with $k \in \mathbb{N}$ large, such that $\omega = \partial f$ satisfies the following: in complex local coordinates z near p_j , one has $\partial_z^\ell \theta(p_j) = \partial_z^\ell \omega(p_j)$ for $\ell = 0, 1, 2$, where $\theta = \theta(z)dz$ and $\omega = \omega(z)dz$.*

Proof. This is a direct consequence of Lemma 2.3. \square

Applying this to the form $\partial G(aV)$ and using the observation we made above, we can construct a C^k holomorphic form ω such that in local coordinates z centered at a critical point p' of Φ (i.e $p' = \{z = 0\}$ in this coordinate), we have for $b = \partial G(aV) - \omega = b(z)dz$

$$(12) \quad \begin{aligned} |\partial_{\bar{z}}^m \partial_z^\ell b(z)| &= O(|z|^{2+\alpha-\ell-m}), \text{ for } \ell + m \leq 2, \quad \text{if } p' \neq p \\ |b(z)| &= O(|z|), \quad \text{if } p' = p. \end{aligned}$$

Now, we let $\chi_1 \in C_0^\infty(M_0)$ be a cutoff function supported in a small neighbourhood U_p of the critical point p and identically 1 near p , and $\chi \in C_0^\infty(M_0)$ is defined similarly with $\chi = 1$ on the support of χ_1 . We will construct $r_1 = r_{11} + hr_{12}$ in two steps : first, we will construct r_{11} to solve equation (11) locally near the critical point p of Φ and then we will construct the global correction term r_{12} away from p by using the extra vanishing of b in (12) at the other critical points.

We define locally in complex coordinates centered at p and containing the support of χ

$$r_{11} := \chi e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 b)$$

where $Rf(z) := -(2\pi i)^{-1} \int_{\mathbb{R}^2} \frac{1}{\bar{z}-\xi} f d\bar{\xi} \wedge d\xi$ for $f \in L^\infty$ compactly supported is the classical Cauchy-Riemann operator inverting locally ∂_z (r_{11} is extended by 0 outside the neighbourhood of p). The function r_{11} is in $C^{3+\alpha}(M_0)$ and we have

$$(13) \quad \begin{aligned} e^{-2i\psi/h} \partial(e^{2i\psi/h} r_{11}) &= \chi_1 (-\partial G(aV) + \omega) + \eta \\ \text{with } \eta &:= e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 b) \partial \chi. \end{aligned}$$

We then construct r_{12} by observing that b vanishes to order $2+\alpha$ at critical points of Φ other than p (from (12)), and $\partial \chi = 0$ in a neighbourhood of any critical point of ψ , so we can find r_{12} satisfying

$$2ir_{12}\partial\psi = (1 - \chi_1)b.$$

This is possible since both $\partial\psi$ and the right hand side are valued in $T_{1,0}^*M_0$, $\partial\psi$ has finitely many isolated 0 on M_0 : r_{12} is then a function which is in $C^{2,\alpha}(M_0 \setminus P)$ where $P := \{p_1, \dots, p_N\}$ is the set of critical points other than p , it extends to a $C^{1,\alpha}(M_0)$ and it satisfies in local complex coordinates z near each p_j

$$|\partial_{\bar{z}}^\beta \partial_z^\gamma r_{12}(z)| \leq C|z - p_j|^{1+\alpha-\beta-\gamma}, \quad \beta + \gamma \leq 2.$$

by using also the fact that $\partial\psi$ can be locally be considered as holomorphic function with a zero of order 1 at each p_j . This implies that $r_1 \in H^2(M_0)$ and we have

$$e^{-2i\psi/h} \partial(e^{2i\psi/h} r_1) = b + h\partial r_{12} = -\partial G(aV) - \omega + h\partial r_{12} + \eta.$$

Now the first error term $\|\partial r_{12}\|_{H^1(M_0)}$ is bounded by

$$\|\partial r_{12}\|_{H^1(M_0)} \leq C \left(\left\| \frac{(1 - \chi_1)b(z)}{\partial_z \psi(z)} \right\|_{H^2(U_p)} \right) \leq C$$

for some constant C , where we used the fact that $\frac{(1 - \chi_1)b(z)}{\partial_z \psi(z)}$ is in $H^2(U_p)$ and independent of h . To deal with the η term, we need the following

Lemma 4.2. *The following estimates hold true*

$$\|\eta\|_{H^2} = O(|\log h|), \quad \|\eta\|_{H^1} \leq O(h|\log h|), \quad \|r_1\|_{L^2} = O(h), \quad \|r_1 - h\tilde{r}_{12}\|_{L^2} = o(h)$$

where \tilde{r}_{12} solves $2i\tilde{r}_{12}\partial\psi = b$.

Proof. We start by observing that

$$(14) \quad \begin{aligned} \|r_1\|_{L^2} &\leq \left\| \chi e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 b) - h \frac{\chi_1 b}{2\partial_z \psi} \right\|_{L^2(U_p)} + h \|\tilde{r}_{12}\|_{L^2(M_0)} + h \left\| \frac{\eta(z)}{2\partial_z \psi} \right\|_{L^2(U_p)}, \\ \|r_1 - h\tilde{r}_{12}\|_{L^2} &\leq \left\| \chi e^{-2i\psi/h} R(e^{2i\psi/h} \chi_1 b) - h \frac{\chi_1 b}{2\partial_z \psi} \right\|_{L^2(U_p)} + h \left\| \frac{\eta(z)}{2\partial_z \psi} \right\|_{L^2(U_p)} \end{aligned}$$

and we will show that the first and last terms in the right hand sides are $o(h)$ while $h\|\tilde{r}_{12}\|_{L^2(M_0)}$ is $O(h)$. The first term is estimated in Proposition 2.7 of [16], it is a $o(h)$, while the last term is clearly bounded by $Ch\|\eta\|_{L^2}$ and the middle one by Ch for some constant C by using that $\partial\psi$ does not vanish on the support of η and the fact that b vanishes at critical points of ψ . Now are going to estimate the H^2 norms of η . Locally in complex coordinates z centered at p (ie. $p = \{z = 0\}$), we have

$$(15) \quad \eta(z) = -\partial_z \chi(z) e^{-\frac{2i\psi(z)}{h}} \int_{\mathbb{C}} e^{\frac{2i\psi(\xi)}{h}} \frac{1}{\bar{z} - \bar{\xi}} \chi_1(\xi) b(\xi) \frac{d\xi_1 d\xi_2}{\pi}, \quad \xi = \xi_1 + i\xi_2.$$

Since b is $C^{2,\alpha}$ in U , we decompose $b(\xi) = \langle \nabla b(0), \xi \rangle + \tilde{b}(\xi)$ using Taylor formula, so we have $\tilde{b}(0) = \partial_{\xi} \tilde{b}(0) = 0$ and we split the integral (15) with $\langle \nabla b(0), \xi \rangle$ and $\tilde{b}(\xi)$. Since the integrand with the $\langle \nabla b(0), \xi \rangle$ is smooth and compactly supported in ξ (recall that $\chi_1 = 0$ on the support of $\partial_z \chi$), we can apply stationary phase to get that

$$\left| \partial_z \chi(z) e^{-\frac{2i\psi(z)}{h}} \int_{\mathbb{C}} e^{\frac{2i\psi(\xi)}{h}} \frac{1}{\bar{z} - \bar{\xi}} \chi_1(\xi) \langle \nabla b(0), \xi \rangle \frac{d\xi_1 d\xi_2}{\pi} \right| \leq Ch^2$$

uniformly in z . Now set $\tilde{b}_z(\xi) = \partial_z \chi(z) \chi_1(\xi) \tilde{b}(\xi) / (\bar{z} - \bar{\xi})$ which is $C^{2,\alpha}$ in ξ and smooth in z . Let $\theta \in C_0^\infty([0, 1])$ be a cutoff function which is equal to 1 near 0 and set $\theta_h(\xi) := \theta(|\xi|/h)$, then we have by integrating by parts

$$(16) \quad \begin{aligned} \int_{\mathbb{C}} e^{\frac{2i\psi(\xi)}{h}} \tilde{b}_z(\xi) d\xi_1 d\xi_2 &= h^2 \int_{\text{supp}(\chi_1)} e^{\frac{2i\psi(\xi)}{h}} \partial_{\bar{\xi}} \left(\frac{1 - \theta_h(\xi)}{2i\partial_{\bar{\xi}} \psi} \partial_{\xi} \left(\frac{\tilde{b}_z(\xi)}{2i\partial_{\xi} \psi} \right) \right) d\xi_1 d\xi_2 \\ &\quad - h \int_{\text{supp}(\chi_1)} e^{\frac{2i\psi(\xi)}{h}} \theta_h(\xi) \partial_{\xi} \left(\frac{\tilde{b}_z(\xi)}{2i\partial_{\xi} \psi} \right) d\xi_1 d\xi_2. \end{aligned}$$

Using polar coordinates with the fact that $\tilde{b}_z(0) = 0$, it is easy to check that the second term in (16) is bounded uniformly in z by Ch^2 . To deal with the first term, we use $\tilde{b}_z(0) = \partial_{\xi} \tilde{b}_z(0) = \partial_{\bar{\xi}} \tilde{b}_z(0) = 0$ and a straightforward computation in polar coordinates shows that the first term of (16) is bounded uniformly in z by $Ch^2 |\log(h)|$. We conclude that

$$\|\eta\|_{L^2} \leq C\|\eta\|_{L^\infty} \leq Ch^2 |\log h|.$$

It is also direct to see that the same estimates holds with a loss of h^{-2} for any derivatives in z, \bar{z} of order less or equal to 2, since they only hit the $\chi(z)$ factor, the $(\bar{z} - \bar{\xi})^{-1}$ factor or the oscillating term $e^{-2i\psi(z)/h}$. So we deduce that

$$\|\eta\|_{H^2} = O(|\log h|).$$

and this ends the proof. \square

We summarize the result of this section with the following

Lemma 4.3. *Let $k \in \mathbb{N}$ be large and $\Phi \in C^k(M_0)$ be a holomorphic function on M_0 which is Morse in M_0 with a critical point at $p \in \text{int}(M_0)$. Let $a \in C^k(M_0)$ be a holomorphic function on M_0 vanishing to high order at any critical point of Φ other than p . Then there exists $r_1 \in H^2(M_0)$ such that $\|r_1\|_{L^2} = O(h)$ and*

$$e^{-\Phi/h}(\Delta + V)e^{\Phi/h}(a + r_1) = O_{L^2}(h|\log h|).$$

4.2. Construction of a_0 . We have constructed the correction terms r_1 which solves the Schrödinger equation to order h as stated in Lemma 4.3. In this subsection, we will construct a holomorphic function a_0 which annihilates the boundary value of the solution on Γ_0 . In particular, we have the following

Lemma 4.4. *There exists a holomorphic function $a_0 \in H^2(M_0)$ independent of h such that*

$$e^{-\Phi/h}(\Delta + V)e^{\Phi/h}(a + r_1 + ha_0) = O_{L^2}(h|\log h|)$$

and

$$[e^{\Phi/h}(a + r_1 + ha_0) + \overline{e^{\Phi/h}(a + r_1 + ha_0)}]_{\Gamma_0} = 0.$$

Proof. First, notice that $h^{-1}r_1|_{\partial M_0} = \tilde{r}_{12}|_{\partial M_0} \in H^{3/2}(\partial M_0)$ is independent of h . Since Φ is purely real on Γ_0 and a is purely imaginary on Γ_0 , we see that this Lemma amounts to construct a holomorphic function $a_0 \in H^2(M_0)$ with the boundary condition

$$\text{Re}(\tilde{r}_{12}) + \text{Re}(a_0) = 0 \text{ on } \Gamma_0.$$

To construct a_0 , it suffices to use (ii) in Corollary 2.2. \square

4.3. Construction of r_2 . The goal of this section is to complete the construction of the complex geometric optic solutions by the following proposition:

Proposition 4.1. *There exist solutions to $(\Delta + V)u = 0$ with boundary condition $u|_{\Gamma_0} = 0$ of the form (10) with r_1, a_0 constructed in the previous sections and r_2 satisfying $\|r_2\|_{L^2} = O(h^{3/2}|\log h|)$.*

This is a consequence of the following Lemma (which follows from the Carleman estimate obtained above):

Lemma 4.5. *If $V \in L^\infty(M_0)$ and $f \in L^2(M_0)$, then for all $h > 0$ small enough, there exists a solution $v \in L^2$ to the boundary value problem*

$$e^{\varphi/h}(\Delta_g + V)e^{-\varphi/h}v = f, \quad v|_{\Gamma_0} = 0,$$

satisfying the estimate

$$\|v\|_{L^2} \leq Ch^{\frac{1}{2}}\|f\|_{L^2}.$$

Proof. The proof is the same as Proposition 2.2 of [16], we repeat the argument for the convenience of the reader. Define for all $h > 0$ the real vector space $\mathcal{A} := \{u \in H_0^1(M_0); (\Delta_g + V)u \in L^2(M_0), \partial_\nu u|_{\Gamma} = 0\}$ equipped with the real scalar product

$$(u, w)_{\mathcal{A}} := \int_{M_0} e^{-2\varphi/h}(\Delta_g u + Vu)(\Delta_g w + Vw) \, dv_g.$$

Observe that since ψ is constant along Γ_0 , $\partial_\nu \varphi = 0$ on Γ_0 . Therefore, we may apply the Carleman estimate of Proposition (3.1) to the weight φ to assert that the space \mathcal{A} is a Hilbert space equipped with the scalar product above. By using the same estimate, the linear functional $L : w \rightarrow \int_{M_0} e^{-\varphi/h} f w \, dv_g$ on \mathcal{A} is continuous and its norm is bounded by $h^{\frac{1}{2}} \|f\|_{L^2}$. By Riesz theorem, there is an element $u \in \mathcal{A}$ such that $(\cdot, u)_{\mathcal{A}} = L$ and with norm bounded by the norm of L . It remains to take $v := e^{-\varphi/h}(\Delta_g u + V u)$ which solves $(\Delta_g + V)e^{-\varphi/h} v = e^{-\varphi/h} f$ and which in addition satisfies the desired norm estimate. Furthermore, since

$$\int_{M_0} e^{-\varphi/h} v (\Delta_g + V) w \, dv_g = \int_{M_0} e^{-\varphi/h} f w \, dv_g$$

for all $w \in \mathcal{A}$, we have by Green's theorem

$$\int_{\partial M_0} e^{-\varphi/h} v \partial_\nu w \, dv_g = 0 = \int_{\Gamma_0} e^{-\varphi/h} v \partial_\nu w \, dv_g$$

for all $w \in \mathcal{A}$. This implies $v = 0$ on Γ_0 . \square

Proof of Proposition 4.1. We note that

$$(\Delta + V)(e^{\Phi/h}(a + r_1 + ha_0) + \overline{e^{\Phi/h}(a + r_1 + ha_0)} + e^{\varphi/h} r_2) = 0$$

if and only if

$$e^{-\varphi/h}(\Delta + V)e^{\varphi/h} r_2 = -e^{-\varphi/h}(\Delta + V)(e^{\Phi/h}(a + r_1 + ha_0) + \overline{e^{\Phi/h}(a + r_1 + ha_0)}).$$

By Lemma 4.4, the right hand side of the above equation is $O_{L^2}(h|\log h|)$. Therefore, using Lemma 4.5 one can find such r_2 which satisfies

$$\|r_2\|_{L^2} \leq Ch^{3/2}|\log h|, \quad r_2|_{\Gamma_0} = 0.$$

Since the ansatz $e^{\Phi/h}(a + r_1 + ha_0) + \overline{e^{\Phi/h}(a + r_1 + ha_0)}$ is arranged to vanish on Γ_0 , the solution

$$u = e^{\Phi/h}(a + r_1 + ha_0) + \overline{e^{\Phi/h}(a + r_1 + ha_0)} + e^{\varphi/h} r_2$$

vanishes on Γ_0 as well. \square

5. IDENTIFYING THE POTENTIAL

We now assume that $V_1, V_2 \in C^{1,\alpha}(M_0)$ are two real valued potentials, with $\alpha > 0$, such that the respective Cauchy data spaces $\mathcal{C}_1^\Gamma, \mathcal{C}_2^\Gamma$ for the operators $\Delta_g + V_1$ and $\Delta_g + V_2$ on $\Gamma \subset \partial M_0$ are equal. Let $\Gamma_0 = \partial M_0 \setminus \Gamma$ be the complement of Γ in ∂M_0 , and possibly by taking Γ slightly smaller, we may assume that Γ_0 contains an open set. Let $p \in M_0$ be an interior point of M_0 such that, using Proposition 2.1, we can choose a holomorphic Morse function $\Phi = \varphi + i\psi$ on M_0 with Φ purely real on Γ_0 , C^k in M_0 for some large $k \in \mathbb{N}$, with a critical point at p . Note that Proposition 2.1 states that we can choose Φ such that none of its critical points on the boundary are degenerate and such that critical points do not accumulate on the boundary.

Proposition 5.1. *If the Cauchy data spaces agree, i.e. if $\mathcal{C}_1^\Gamma = \mathcal{C}_2^\Gamma$, then $V_1(p) = V_2(p)$.*

Proof. Let a be a holomorphic function on M_0 which is purely imaginary on Γ_0 with $a(p) \neq 0$ and $a(p') = 0$ to large order for all other critical point p' of Φ . The existence of a is insured by Lemma 2.3. Let u_1 and u_2 be H^2 solutions on M_0 to

$$(\Delta_g + V_j)u_j = 0$$

constructed in Section 4 with $\Phi = \phi + i\psi$ for Carleman weight for u_1 and $-\Phi$ for u_2 , thus of the form

$$\begin{aligned} u_1 &= e^{\Phi/h}(a + ha_0 + r_1) + \overline{e^{\Phi/h}(a + ha_0 + r_1)} + e^{\varphi/h}r_2 \\ u_2 &= e^{-\Phi/h}(a + hb_0 + s_1) + \overline{e^{-\Phi/h}(a + hb_0 + s_1)} + e^{-\varphi/h}s_2 \end{aligned}$$

and with boundary value $u_j|_{\partial M_0} = f_j$, where f_j vanishes on Γ_0 . We can write by Green formula

$$\begin{aligned} \int_{M_0} u_1(V_1 - V_2)u_2 dv_g &= - \int_{M_0} (\Delta_g u_1 \cdot u_2 - u_1 \cdot \Delta_g u_2) dv_g \\ &= - \int_{\partial M_0} (\partial_\nu u_1 \cdot f_2 - f_1 \cdot \partial_\nu u_2) dv_g. \end{aligned}$$

Since the Cauchy data for $\Delta_g + V_1$ agrees on Γ with that of $\Delta_g + V_2$, there exists a solution v of the boundary value problem

$$(\Delta_g + V_2)v = 0, \quad v|_{\partial M_0} = f_1,$$

satisfying $\partial_\nu v = \partial_\nu u_1$ on Γ . Since $f_j = 0$ on Γ_0 , this implies that

$$\begin{aligned} (17) \quad \int_{M_0} u_1(V_1 - V_2)u_2 dv_g &= - \int_{M_0} (\Delta_g u_1 \cdot u_2 - u_1 \cdot \Delta_g u_2) dv_g = - \int_{\partial M_0} (\partial_\nu u_1 \cdot f_2 - f_1 \cdot \partial_\nu u_2) dv_g \\ &= - \int_{\partial M_0} (\partial_\nu v \cdot f_2 - v \cdot \partial_\nu u_2) dv_g = - \int_{M_0} (\Delta_g v \cdot u_2 - v \cdot \Delta_g u_2) dv_g = 0 \end{aligned}$$

since $\Delta_g + V_2$ annihilates both v and u_2 . We substitute in the full expansion for u_1 and u_2 and, setting $V := V_1 - V_2$, and using the estimates in Lemmas 4.2, 4.1 and 4.4 and, we obtain

$$(18) \quad 0 = I_1 + I_2 + o(h),$$

where

$$(19) \quad I_1 = \int_{M_0} V(a^2 + \bar{a}^2) dv_g + 2\operatorname{Re} \int_{M_0} e^{2i\psi/h} V |a|^2 dv_g,$$

$$(20) \quad I_2 = 2h \operatorname{Re} \int_{M_0} aV \left(e^{2i\psi/h} \left(\frac{\bar{s}_1}{h} + b_0 \right) + e^{-2i\psi/h} \left(a_0 + \frac{\bar{r}_1}{h} \right) + b_0 + a_0 + \frac{s_1 + r_1}{h} \right) dv_g.$$

Remark 5.1. We observe from the last identity in Lemma 4.2 that r_1/h in the expression I_2 can be replaced by the term \tilde{r}_{12} satisfying $2i\tilde{r}_{12}\partial\psi = b$ up to an error which can go in the $o(h)$ in (18), and similarly for the term s_1/h which can be replaced by a term \tilde{s}_{12} independent of h .

We will apply the stationary phase to these two terms in the following two Lemmas.

Lemma 5.2. *The following estimates holds true*

$$I_2 = 2h \operatorname{Re} \left(\int_{M_0} V a(b_0 + a_0 + r_{12} + s_{12} + \tilde{s}_{12} + \tilde{r}_{12}) dv_g \right) + o(h).$$

where r_{12} , s_{12} , \tilde{r}_{12} and \tilde{s}_{12} are independent of h .

Proof. We start by the following

Lemma 5.3. *Let $f \in L^1(M_0)$, then as $h \rightarrow 0$*

$$\int_{M_0} e^{2i\psi/h} f dv_g = o(1).$$

Proof. Since $C^k(M_0)$ is dense in $L^1(M_0)$ for all $k \in \mathbb{N}$, it suffices to prove the Lemma for $f \in C^k(M_0)$. Let $\epsilon > 0$ be small, and choose cut off function χ which is identically equal to 1 on the boundary such that

$$\int_{M_0} \chi |f| dv_g \leq \epsilon.$$

Then, splitting the integral and using stationary phase for the $1 - \chi$ term, we obtain

$$\left| \int_{M_0} e^{2i\psi/h} f dv_g \right| \leq \left| \int_{M_0} (1 - \chi) e^{2i\psi/h} f dv_g \right| + \left| \int_{M_0} \chi e^{2i\psi/h} f dv_g \right| \leq \epsilon + O_\epsilon(h)$$

which concludes the proof by taking h small enough depending on ϵ . □

The proof of Lemma 5.2 is a direct consequence of Lemma 5.3 and Remark 5.1. □

The second Lemma will be proved in the end of this section.

Lemma 5.4. *The following estimate holds true*

$$I_1 = \int_{M_0} V(a^2 + \bar{a}^2) dv_g + h C_p V(p) |a(p)|^2 \operatorname{Re}(e^{2i\psi(p)/h}) + o(h)$$

with $C_p \neq 0$ and independent of h .

With these two Lemmas, we can write (18) as

$$0 = \int_{M_0} V(a^2 + \bar{a}^2) dv_g + O(h)$$

and thus we can conclude that

$$0 = \int_{M_0} V(a^2 + \bar{a}^2) dv_g.$$

Therefore, (18) becomes

$$0 = C_p V(p) |a(p)|^2 \operatorname{Re}(e^{2i\psi(p)/h}) + 2 \operatorname{Re} \left(\int_{M_0} V a(b_0 + s_1 + a_0 + r_1) dv_g \right) + o(1).$$

Since $\psi(p) \neq 0$ we may choose a sequence of $h_j \rightarrow 0$ such that $\operatorname{Re}(e^{2i\psi(p)/h_j}) = 1$ and another sequence $\tilde{h}_j \rightarrow 0$ such that $\operatorname{Re}(e^{2i\psi(p)/\tilde{h}_j}) = -1$ for all j . Adding the expansion with $h = h_j$ and $h = \tilde{h}_j$, we deduce that

$$0 = 2C_p V(p) |a(p)|^2 + o(1)$$

as $j \rightarrow \infty$, and since $C_p \neq 0$, $a(p) \neq 0$, we conclude that $V(p) = 0$. The set of $p \in M_0$ for which we can conclude this is dense in M_0 by Proposition 2.1. Therefore we can conclude

that $V(p) = 0$ for all $p \in M_0$. □

We now prove Lemma 5.4.

Proof of Lemma 5.4. Let χ be a smooth cutoff function on M_0 which is identically 1 everywhere except outside a small ball containing p and no other critical point of Φ , and $\chi = 0$ near p . We split the oscillatory integral in two parts:

$$\begin{aligned} \int_{M_0} (e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g &= \int_{M_0} \chi (e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g \\ &\quad + \int_{M_0} (1 - \chi) (e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g \end{aligned}$$

The phase ψ has nondegenerate critical points, therefore, a standard application of the stationary phase at p gives

$$\int_{M_0} (1 - \chi) (e^{2i\psi/h} + e^{-2i\psi/h}) V(p) |a|^2 dv_g = h C_p |a(p)|^2 V(p) \operatorname{Re}(e^{2i\psi(p)/h}) + o(h)$$

where C_p is a non-zero number which depends on the Hessian of ψ at the point p . Define the potential $\tilde{V}(\cdot) := V(\cdot) - V(p) \in C^{1,\alpha}(M_0)$, then we show that

$$(21) \quad \int_{M_0} (1 - \chi) (e^{2i\psi/h} + e^{-2i\psi/h}) \tilde{V} |a|^2 dv_g = o(h).$$

Indeed, first by integration by parts and using $\Delta_g \psi = 0$ one has

$$\begin{aligned} \int_{M_0} (1 - \chi) (e^{2i\psi/h} + e^{-2i\psi/h}) \tilde{V} |a|^2 dv_g &= \frac{h}{2i} \int_{M_0} \langle d(e^{2i\psi/h} - e^{-2i\psi/h}), d\psi \rangle \tilde{V} \frac{(1 - \chi) |a|^2}{|d\psi|^2} dv_g \\ &= \frac{h}{2i} \int_{M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) \langle d\left(\frac{(1 - \chi) |a|^2 \tilde{V}}{|d\psi|^2}\right), d\psi \rangle dv_g \end{aligned}$$

but we can see that $\langle d((1 - \chi) |a|^2 \tilde{V} / |d\psi|^2), d\psi \rangle \in L^1(M_0)$: this follows directly from the fact that \tilde{V} is in the Hölder space $C^{1,\alpha}(M_0)$ and $\tilde{V}(p) = 0$, and from the non degeneracy of $\operatorname{Hess}(\psi)$. It then suffice to use Lemma 5.3 to conclude that (21) holds. Using similar argument, we now show that

$$\int_{M_0} \chi (e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g = o(h).$$

Indeed, since a vanishes to large order at all boundary critical points of ψ , we may write

$$\begin{aligned} \int_{M_0} \chi (e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g &= \frac{h}{2i} \int_{M_0} \langle d(e^{2i\psi/h} - e^{-2i\psi/h}), d\psi \rangle V \frac{\chi |a|^2}{|d\psi|^2} dv_g \\ &= -\frac{h}{2i} \int_{M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) \operatorname{div}_g \left(V \frac{\chi |a|^2}{|d\psi|^2} \nabla^g \psi \right) dv_g \\ &\quad + \frac{h}{2i} \int_{\partial M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) V \frac{|a|^2}{|d\psi|^2} \partial_\nu \psi dv_g. \end{aligned}$$

For the interior integral we use Lemma 5.3 to conclude that

$$-\frac{h}{2i} \int_{M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) \operatorname{div}_g \left(V \frac{\chi |a|^2}{|d\psi|^2} \nabla^g \psi \right) dv_g = o(h)$$

and for the boundary integral, we write $\partial M_0 = \Gamma_0 \cup \Gamma$ and observe that on Γ_0 , $\psi = 0$ so $(e^{2i\psi/h} - e^{-2i\psi/h}) = 0$, while on Γ we have $V = 0$ from the boundary determinacy proved in Proposition 7.1 of the Appendix. Therefore

$$\int_{M_0} \chi(e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g = o(h)$$

and the proof is complete. \square

6. INVERSE SCATTERING

We first obtain, as a trivial consequence of Theorem 1.1, a result about inverse scattering for asymptotically hyperbolic surface (AH in short). Recall that an AH surface is an open complete Riemannian surface (X, g) such that X is the interior of a smooth compact surface with boundary \bar{X} , and for any smooth boundary defining function x of $\partial\bar{X}$, $\bar{g} := x^2g$ extends as a smooth metric to \bar{X} , with curvature tending to -1 at $\partial\bar{X}$. If $V \in C^\infty(\bar{X})$ and $V = O(x^2)$, then we can define a scattering map as follows (see for instance [17, 12] or [13]): first the L^2 kernel $\ker_{L^2}(\Delta_g + V)$ is a finite dimensional subspace of $xC^\infty(\bar{X})$ and in one-to-one correspondence with $E := \{(\partial_x \psi)|_{\partial\bar{X}}; \psi \in \ker_{L^2}(\Delta_g + V)\}$ where $\partial_x := \nabla^{\bar{g}}x$ is the normal vector field to $\partial\bar{X}$ for \bar{g} , then for $f \in C^\infty(\partial\bar{X})$, there exists a function $u \in C^\infty(\bar{X})$, unique modulo $\ker_{L^2}(\Delta_g + V)$, such that $(\Delta_g + V)u = 0$ and $u|_{\partial\bar{X}} = f$. Then one can see that the scattering map $\mathcal{S} : C^\infty(\partial\bar{X}) \rightarrow C^\infty(\partial\bar{X})/E$ is defined by $\mathcal{S}f := \partial_x u|_{\partial\bar{X}}$. We thus obtain

Corollary 6.1. *Let (X, g) be an asymptotically hyperbolic manifold and let $V_1, V_2 \in x^2C^\infty(\bar{X})$ be two potentials and $\Gamma \subset \partial\bar{X}$ an open subset of the conformal boundary. Assume that*

$$\{\partial_x u|_{\partial\bar{X}}; u \in \ker_{L^2}(\Delta_g + V_1)\} = \{\partial_x u|_{\partial\bar{X}}; u \in \ker_{L^2}(\Delta_g + V_2)\}$$

and let \mathcal{S}_j be the scattering map for the operator $\Delta_g + V_j$ for $j = 1, 2$. If $\mathcal{S}_1 f = \mathcal{S}_2 f$ on Γ for all $f \in C_0^\infty(\Gamma)$ then $V_1 = V_2$.

Proof. Let x be a smooth boundary defining function of $\partial\bar{X}$, and let $\bar{g} = x^2g$ be the compactified metric and define $\bar{V}_j := V_j/x^2 \in C^\infty(\bar{X})$. By conformal invariance of the Laplacian in dimension 2, one has

$$\Delta_g + V_j = x^2(\Delta_{\bar{g}} + \bar{V}_j)$$

and so if $\ker_{L^2}(\Delta_g + V_1) = \ker_{L^2}(\Delta_g + V_2)$ and $\mathcal{S}_1 = \mathcal{S}_2$ on Γ , then the Cauchy data spaces \mathcal{C}_i^Γ for the operator $\Delta_{\bar{g}} + \bar{V}_j$ are the same. Then it suffices to apply the result in Theorem 1.1. \square

Next we consider the asymptotically Euclidean scattering at 0 frequency. An asymptotically Euclidean surface is a non-compact Riemann surface (X, g) , which compactifies into \bar{X} and such that the metric in a collar $(0, \epsilon)_x \times \partial\bar{X}$ near the boundary is of the form

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2}$$

where $h(x)$ is a smooth one-parameter family of metrics on $\partial\bar{X}$ with $h(0) = d\theta_{S^1}^2$ is the metric with length 2π on each copy of S^1 that forms the connected components of $\partial\bar{X}$. Notice that using the coordinates $r := 1/x$, g is asymptotic to $dr^2 + r^2 d\theta_{S^1}^2$ near $r \rightarrow \infty$. A particular case is given by the surfaces with Euclidean ends, ie. ends isometric to $\mathbb{R}^2 \setminus B(0, R)$ where

$B(0, R) = \{z \in \mathbb{R}^2; |z| \geq R\}$. Note that g is conformal to an asymptotically cylindrical metric, or 'b-metric' in the sense of Melrose [24],

$$g_b := x^2 g = \frac{dx^2}{x^2} + h(x)$$

and the Laplacian satisfies $\Delta_g = x^2 \Delta_{g_b}$. Each end of X is of the form $(0, \epsilon)_x \times S_\theta^1$ and the operator Δ_{g_b} has the expression in the ends

$$\Delta_{g_b} = -(x\partial_x)^2 + \Delta_{\partial\bar{X}} + xP(x, \theta; x\partial_x, \partial_\theta)$$

for some smooth differential operator $P(x, \theta; x\partial_x, \partial_\theta)$ in the vector fields $x\partial_x, \partial_\theta$ down to $x = 0$. Let us define $V_b := x^{-2}V$, which is compactly supported and

$$H_b^{2m} := \{u \in L^2(X, \text{dvol}_{g_b}); \Delta_{g_b}^m u \in L^2(X, \text{dvol}_{g_b})\}, \quad m \in \mathbb{N}_0.$$

We also define the following spaces for $\alpha \in \mathbb{R}$

$$F_\alpha := \ker_{x^\alpha H_b^2}(\Delta_{g_b} + V_b).$$

Since the eigenvalues of Δ_{S^1} are $\{j^2; j \in \mathbb{N}_0\}$, the relative Index theorem of Melrose [24, Section 6.2] shows that $\Delta_{g_b} + V_b$ is Fredholm from $x^\alpha H_b^2$ to $x^\alpha H_b^0$ if $\alpha \notin \mathbb{Z}$. Moreover, from subsection 2.2.4 of [24], we have that any solution of $(\Delta_{g_b} + V_b)u = 0$ in $x^\alpha H_b^2$ has an asymptotic expansion of the form

$$u \sim \sum_{j > \alpha, j \in \mathbb{Z}} \sum_{\ell=0}^{\ell_j} x^j (\log x)^\ell u_{j,\ell}(\theta), \quad \text{as } x \rightarrow 0$$

for some sequence $(\ell_j)_j$ of non negative integers and some smooth function $u_{j,\ell}$ on S^1 . In particular, it is easy to check that $\ker_{L^2(X, \text{dvol}_g)}(\Delta_g + V) = F_{1+\epsilon}$ for $\epsilon \in (0, 1)$.

Theorem 6.2. *Let (X, g) be an asymptotically Euclidean surface and V_1, V_2 be two compactly supported smooth potentials and x be a boundary defining function. Let $\epsilon \in (0, 1)$ and assume that for any $j \in \mathbb{Z}$ and any function $\psi \in \ker_{x^{j-\epsilon} H_b^2}(\Delta_g + V_1)$ there is a $\varphi \in \ker_{x^{j-\epsilon} H_b^2}(\Delta_g + V_2)$ such that $\psi - \varphi = O(x^\infty)$, and conversely. Then $V_1 = V_2$.*

Proof. The idea is to reduce the problem to the compact case. First we notice that by unique continuation, $\psi = \varphi$ where $V_1 = V_2 = 0$. Now it remains to prove that, if R_η denote the restriction of smooth functions on X to $\{x \geq \eta\}$ and V is a smooth compactly supported potential in $\{x \geq \eta\}$, then the set $\cup_{j=0}^\infty R_\eta(F_{-j-\epsilon})$ is dense in the set N_V of $H^2(\{x \geq \eta\})$ solutions of $(\Delta_g + V)u = 0$. The proof is well known for positive frequency scattering (see for instance Lemma 3.2 in [25]), here it is very similar so we do not give much details. The main argument is to show that it converges in L^2 sense and then use elliptic regularity; the L^2 convergence can be shown as follows: let $f \in N_V$ such that

$$\int_{x \geq \eta} f \psi \text{dvol}_g = 0, \quad \forall \psi \in \cup_{j=0}^\infty F_{-j-\epsilon},$$

then we want to show that $f = 0$. By Proposition 5.64 in [24], there exists $k \in \mathbb{N}$ and a generalized right inverse G_b for $P_b = \Delta_{g_b} + V_b$ (here, as before, $x^2 V_b = V$) in $x^{-k-\epsilon} H_b^2$, such that $P_b G_b = \text{Id}$. This holds in $x^{-k-\epsilon} H_b^2$ for k large enough since the cokernel of P_b on this space becomes 0 for k large. Let $\omega = G_b f$ so that $(\Delta_{g_b} + V_b)\omega = f$, and in particular this function is 0 in $\{x < \eta\}$. The asymptotic behaviour of the integral kernel $G_b(z, z')$ of G_b as

$z \rightarrow \infty$ is given in Proposition 5.64 of [24] uniformly in $z' \in \{x \geq \eta\}$, we have for all $J \in \mathbb{N}$ and using the radial coordinates $(x, \theta) \in (0, \epsilon) \times S^1$ for z in the ends

$$G_b(z, z') = \sum_{j=-k}^J \sum_{\ell=0}^{\ell_j} x^j (\log x)^\ell \psi_j(\theta, z') + o(x^J)$$

for some functions $\psi_{j,\ell} \in x^{k-j-\epsilon} H_b^2$ and some sequence $(\ell_j)_j$ of non-negative integers. But the fact that $(\Delta_{g_b} + V_b)G_b(z, z') = \delta(z - z')$ as distributions implies directly that $(\Delta_{g_b} + V_b)\psi_j(\theta, \cdot) = 0$. Using our assumption on f , we deduce that $\int_X \psi_j(\theta, z') f(z') d\text{vol}_{g_b} = 0$ for all $j \in \mathbb{N}_0$ and so the function ω vanish faster than all power of x at infinity. Then by unique continuation, we deduce that $\omega = 0$ in $\{x \leq \epsilon\}$. Since now $\omega \in H^2$, its Cauchy data at $x = \eta$ are 0 and $\Delta_{g_b} + V_b$ is self adjoint for the measure $d\text{vol}_{g_b}$, we can use the Green formula to obtain

$$\int_{x \geq \eta} |f|^2 d\text{vol}_{g_b} = \int_{x \geq \eta} \omega(\Delta_{g_b} + V_b) \bar{f} d\text{vol}_{g_b} = 0.$$

The H^2 density is easy using elliptic regularity. \square

7. APPENDIX : BOUNDARY DETERMINATION

In this appendix, we give a short proof of the fact that the partial Cauchy data space on $\Gamma \subset \partial M$ determines the potential on Γ when the potential is in $C^{0,\alpha}(M)$ for some $\alpha \in (0, 1)$. This result is shown for the conductivity problem on a domain of \mathbb{R}^n in [19] under the assumption that the conductivity has roughly $n/2$ -derivatives, it is also shown in [29] for continuous potentials on a smooth domain of \mathbb{R}^n by using a limiting argument from the smooth case (which they analyze using micolocal analysis near the boundary). Alessandrini [1] also proved such a result under the assumption that the domain is Lipschitz and the coefficients of the operator are in $W^{1,p}$ for $p > n$, while Brown [5] studied the case of Lipschitz domains with a continuous conductivity. Since the result in our setting is not explicitly written down, but certainly known from specialists, we provide a short proof without too many details, and using the approach of [5]. We shall prove

Proposition 7.1. *Let $\Gamma \subset \partial M_0$ be an non-empty open subset of the boundary. If $V_1, V_2 \in C^{0,\alpha}(M)$ for some $\alpha > 0$ and their associated Cauchy data spaces $\mathcal{C}_1^\Gamma, \mathcal{C}_2^\Gamma$ defined in (1) are equal, then $V_1|_\Gamma = V_2|_\Gamma$.*

The key to proving this proposition is the existence of solutions to $(\Delta_g + V_i)u = 0$ which concentrate near a point $p \in \Gamma$. First we need a solvability result for the equation $(\Delta_g + V_i)u = f$, which is an easy consequence of the Carleman estimate of Proposition 3.1, and follows the method of Salo-Tzou [27, Section 6]. If we fix $h > 0$ small and take $\varphi = 1$ in the Carleman estimate of Proposition 3.1, we obtain easily that there is a constant C such that for all functions in $H^2(M_0)$ satisfying $u|_{\partial M_0} = 0$

$$(22) \quad \|u\|_{H^2}^2 + \|\partial_\nu u\|_{L^2(\Gamma_0)}^2 \leq C(\|(\Delta + V_i)u\|_{L^2}^2 + \|\partial_\nu u\|_{L^2(\Gamma)}^2)$$

As a consequence, we deduce the following solvability result: let

$$\mathcal{B} := \{w \in H^2(M_0) \cap H_0^1(M_0) \mid \partial_\nu w|_\Gamma = 0\}$$

be the closed subspace of $H^2(M_0)$ under the H^2 norm and let \mathcal{B}^* be its dual space then

Corollary 7.1. *Let $i = 1, 2$, then for all $f \in L^2(M_0)$ there exists $u \in H^2(M_0)$ solving the equation*

$$(\Delta_g + V_i)u = f$$

with boundary condition $u|_{\Gamma_0} = 0$, and $\|u\|_{L^2} \leq C\|f\|_{\mathcal{B}^}$.*

Proof. Set $\mathcal{A} := \{w \in H_0^1(M_0) \mid (\Delta_g + V_i)w \in L^2, \partial_\nu w|_\Gamma = 0\}$ equipped with the inner product

$$(v, w)_\mathcal{A} := \int_{M_0} (\Delta_g + V_i)v(\Delta_g + V_i)\bar{w} \, dv_g.$$

Thanks to (22), \mathcal{A} is a Hilbert space and $\mathcal{A} = \mathcal{B}$. For each $f \in L^2(M_0)$, let us define the linear functional on \mathcal{B}

$$L_f : w \mapsto \int_{M_0} wf \, dv_g.$$

By (22), we have that for all $w \in \mathcal{B}$

$$|L_f(w)| \leq \|f\|_{\mathcal{B}^*} \|w\|_\mathcal{B} \leq \|f\|_{\mathcal{B}^*} \|w\|_\mathcal{A}.$$

Therefore, by Riesz Theorem, there exists $v_f \in \mathcal{A}$ such that

$$\int_{M_0} (\Delta_g + V_i)\overline{v_f}(\Delta_g + V_i)w \, dv_g = \int_{M_0} wf \, dv_g$$

for all $w \in \mathcal{B}$. Furthermore, $\|(\Delta_g + V_i)v_f\|_{L^2} \leq \|f\|_{\mathcal{B}^*}$. Setting $u := (\Delta_g + V_i)\overline{v_f}$ we have that $(\Delta_g + V_i)u = f$ and $\|u\|_{L^2} \leq \|f\|_{\mathcal{B}^*}$. To obtain the boundary condition for u , observe that since

$$\int_{M_0} u(\Delta_g + V_i)w \, dv_g = \int_{M_0} fw \, dv_g$$

for all $w \in \mathcal{B}$, by Green's theorem

$$\int_{M_0} u\partial_\nu w \, dv_g = 0 = \int_{\Gamma_0} u\partial_\nu w \, dv_g$$

for all $w \in \mathcal{B}$. This implies $u = 0$ on Γ_0 . \square

Clearly, it suffices to assume that Γ is a small piece of the boundary which is contained in a single coordinate chart with complex coordinates $z = x + iy$ where $|z| \leq 1$, $\text{Im}(z) > 0$ and the boundary is given by $\{y = 0\}$. Moreover the metric is of the form $e^{2\rho}|dz|^2$ for some smooth function ρ . Let $p \in \Gamma$ and possibly by translating the coordinates, we can assume that $p = \{z = 0\}$. Let $\eta \in C^\infty(M_0)$ be a cutoff function supported in a small neighbourhood of p . For $h > 0$ small, we define the smooth function $v_h \in C^\infty(M_0)$ supported near p via the coordinate chart $Z = (x, y) \in \mathbb{R}^2$ by

$$(23) \quad v_h(Z) := \eta(Z/\sqrt{h})e^{\frac{1}{h}\alpha \cdot Z}$$

where $\alpha := (i, -1) \in \mathbb{C}^2$ is chosen such that $\alpha \cdot \alpha = 0$. We thus get $(\partial_x^2 + \partial_y^2)e^{\alpha \cdot Z} = 0$ and thus $\Delta_g e^{\alpha \cdot Z} = 0$ by conformal covariance of the Laplacian. Therefore, we have in local coordinates

$$(24) \quad \Delta_g v_h(Z) = \frac{1}{h}e^{\frac{1}{h}\alpha \cdot Z}(\Delta_g \eta)\left(\frac{Z}{\sqrt{h}}\right) + \frac{2}{h^{3/2}}e^{\frac{1}{h}\alpha \cdot Z}\langle d\eta\left(\frac{Z}{\sqrt{h}}\right), \alpha \cdot dZ \rangle_g.$$

Lemma 7.2. *If $V \in C^{0,\alpha}(M_0)$ for $q > 2$, then there exists a solution $u_h \in H^2$ to $(\Delta_g + V)u = 0$ of the form*

$$u_h = v_h + R_h,$$

with v_h defined in (23) and $\|R_h\|_{L^2} \leq Ch^{5/4}$, satisfying $\text{supp}(R_h|_{\partial M_0}) \subset \Gamma$.

Proof of Lemma 7.2. We need to find R_h satisfying $\|R_h\|_{L^2} \leq Ch^{5/4}$ and solving

$$(\Delta_g + V)R_h = -(\Delta_g + V)v_h =: M_h.$$

Thanks to Corollary 7.1, it suffices to show that $\|M_h\|_{\mathcal{B}^*} \leq Ch^{5/4}$. Thus, let $w \in \mathcal{B}$, then we have by (24)

$$\int_{M_0} w M_h \, dv_g = I_1 + I_2 + I_3$$

where

$$I_1 := \int_{|Z| \leq \sqrt{h}} w V_i v_h e^{2\rho} dZ, \quad I_2 := \frac{1}{h} \int_{|Z| \leq \sqrt{h}} w e^{\frac{1}{h}\alpha \cdot Z} \chi_1\left(\frac{Z}{\sqrt{h}}\right) e^{2\rho} dZ,$$

$$I_3 := \frac{2}{h^{3/2}} \int_{|Z| \leq \sqrt{h}} w \chi_2\left(\frac{Z}{\sqrt{h}}\right) e^{\frac{1}{h}\alpha \cdot Z} dZ$$

and $\chi_1 = \Delta_g \eta$, $\chi_2 = i\partial_x \eta - \partial_y \eta$. In the above equation the third term I_3 has the worst growth when $h \rightarrow 0$. We will analyze its behavior and the preceding terms can be treated in similar fashion. One has

$$\begin{aligned} I_3 &= -h^{1/2} \int_{|Z| \leq \sqrt{h}} w \chi_2\left(\frac{Z}{\sqrt{h}}\right) (\partial_x - i\partial_y)^2 e^{\frac{1}{h}\alpha \cdot Z} dZ \\ &= -h^{1/2} \int_{|Z| \leq \sqrt{h}} (\partial_x - i\partial_y)^2 (w \chi_2\left(\frac{Z}{\sqrt{h}}\right)) e^{\frac{1}{h}\alpha \cdot Z} dZ. \end{aligned}$$

Notice that the boundary term in the integration by parts vanishes because $w \in H_0^1$ and $\partial_\nu w|_{\partial M_0}$ vanishes on the support of η . The term $(\partial_x - i\partial_y)^2 (w \chi_2(\frac{Z}{\sqrt{h}}))$ has derivatives hitting both $\chi_2(\frac{Z}{\sqrt{h}})$ and w . The worst growth in h would occur when both derivatives hit $\chi_2(\frac{Z}{\sqrt{h}})$ in which case a h^{-1} factor would come out. Combined with the $h^{1/2}$ term in front of the integral this gives a total of a $h^{-1/2}$ in front. By this observation we have improved the growth from $h^{-3/2}$ to $h^{-1/2}$. Repeating this line of argument and using Cauchy-Schwarz inequality, we can see that $|I_3| \leq Ch^{5/4} \|w\|_{H^2}$ (an elementary computation shows that functions of the form $\chi(Z/\sqrt{h})e^{\frac{1}{h}\alpha \cdot Z}$ have L^2 norm bounded by $Ch^{3/4}$). Therefore, $\|\frac{1}{h^{3/2}} \chi_2(\frac{Z}{\sqrt{h}}) e^{\frac{1}{h}\alpha \cdot Z}\|_{\mathcal{B}'} \leq Ch^{5/4}$ and we are done. \square

Proof of Proposition 7.1. It suffices to plug the solutions u_h^1, u_h^2 from Proposition 7.2 into the boundary integral identity (17). A simple calculation using the fact that $V_1 - V_2$ is in $C^{0,\gamma}(M_0)$ yields that

$$0 = \int_{M_0} u_h^1 (V_1 - V_2) u_h^2 \, dv_g = Ch^{3/2} (V_1(p) - V_2(p)) + o(h^{3/2})$$

and we are done. \square

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